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The Ellsberg paradox: A challenge to quantum decision theory?[☆]

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HIGHLIGHTS

- A simple quantum decision theory model that explains the Ellsberg paradox and is in conformity with the evidence.
- Reviews earlier attempts to explain the Ellsberg paradox, both classical and quantum.
- Provides a simple introduction to quantum decision theory.

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ABSTRACT

We set up a simple quantum decision model of the Ellsberg paradox. We find that the matching probabilities that our model predict are in good agreement with those empirically measured by Dimmock et al. (2015). Our derivation is parameter free. It only depends on quantum probability theory in conjunction with the heuristic of insufficient reason. We suggest that much of what is normally attributed to probability weighting might actually be due to quantum probability.

1. Introduction

Consider the following version of the Ellsberg experiment[1] due to Dimmock, Kouwenberg, and Wakker (2015). This involves two urns: The known urn (K) contains 100 balls of n different colors, $1 < n \leq 100$, the same number of balls for each color (for example, if $n = 5$ then there are 5 different colors and 20 balls of each color in K). The unknown urn (U) also contains n balls of the same colors as urn K but in unknown proportions. The subject is asked to select one of the urns (K or U). A ball is drawn at random from the urn chosen by the subject. There are two versions. In the low probability version, the subject wins a sum of money if the color of the ball drawn matches a preassigned color (which, however, could be chosen by the subject). In the high probability version, the subject wins the sum of money if the color of the randomly drawn ball matches any one of $n-1$ preassigned colors (again, these colors could be chosen by the subject). These two versions are, of course, equivalent if $n = 2$, but different for $n > 2$. The subject is also allowed to declare indifference between K and U.

If a subject prefers K to U, she is called *ambiguity averse*. If she prefers U to K, she is called *ambiguity seeking*. If she is indifferent between K and U she is called *ambiguity neutral*.

Dimmock et al. (2015) perform a second set of experiments. Here, the ratio of the colors (whatever they are) in U were kept fixed. However, the ratio in K was varied until a subject declared indifference. This ratio is then called the *matching probability*. For example, in the low probability treatment, they found that for $n = 10$ colors, subjects (on average) declared indifference between K and U when the new urn K contained 22 balls (out of 100) of the winning color. Hence, the matching probability of 0.1 is $m(0.1) = 0.22 > 0.1$. Thus, subjects exhibited ambiguity seeking for the low probability of 0.1. In the high probability treatment, they found that, again for $n = 10$ colors, subjects (on average) declared indifference when the new urn K contained 69 balls of the winning colors. Hence, the matching probability of 0.9 is $m(0.9) = 0.69 < 0.9$. For $n = 2$ colors, subjects (on average) declared indifference when the new urn K contained 40 balls of the winning color. Hence, $m(0.5) = 0.4 < 0.5$. Thus, subjects exhibited ambiguity aversion for medium and high probabilities but ambiguity seeking for low probabilities.

The reason why preferring K to U (or U to K) was regarded as paradoxical[2] is as follows. Although experimental subjects know

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1 Ellsberg (1961, 2001) and Keynes (1921).

2 This was the situation before the advent of the source method, see Section 3.5.

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the proportion of colors in urn K (it contains exactly the same number of each color), they do not know the ratio in urn U. But they have no reason to believe that one color is more likely than another. Hence, by the heuristic of *insufficient reason* (or *equal a' priori probabilities*),[3] they should assign the same probability to each color in urn U.[4] Hence, they should have no reason to prefer K to U or U to K on probabilistic grounds. Keynes (1921) pointed out that there is a difference in the strength or quality of the evidence. Subjects may reason that, although the assignment of the same probability to each color is sound, they are more confident in the correctness of this judgement in the case of K than in the case of U. Hence, they prefer K to U. Thus, their preference works through the utility channel rather than the probability channel. However, this explanation appears to be contradicted by the evidence of Dimmock et al. (2015) that subjects are ambiguity seeking for low probabilities. Moreover, even when subjects are told that each color in U has the same probability, so that the heuristic of insufficient reason is not needed, they still exhibit a preference for K over U (Rode, Cosmides, Hell, & Tooby, 1999). Furthermore, because the probabilities in urn U have been revealed, the observed choice of K over U cannot be attributed to ambiguity aversion or differences in the strength or quality of the evidence.

The importance of the Ellsberg experiments is twofold. First, they provide tests for competing decision theories. Second, there are many real-world situations that appear similar to the Ellsberg paradox. One example is that of *home-bias* in investment (French & Poterba, 1991; Obstfeld & Rogoff, 2000). Investors are often observed to prefer investing in a domestic asset over a foreign asset with the same return and the same riskiness.

La Mura (2009) proposed to replace standard (Kolmogorov) probabilities in expected utility theory with quantum probabilities; and called the resulting decision theory *projective expected utility theory*. He gave an axiomatic foundation for this new theory and derived the equivalence of the preference representation and the utility representation. He applied the new theory to explain the Allais paradox. He suggested it may explain the Ellsberg paradox.

Busemeyer and Bruza (2012, section 9.1.2) applied projective expected utility theory to explain the Ellsberg paradox. Their model has a free parameter, a. If $a > 0$ we get ambiguity aversion, if $a = 0$, we get ambiguity neutrality, and if $a < 0$ we get ambiguity seeking. However, it cannot explain the simultaneous occurrence in the same subject of ambiguity seeking (for low probabilities), ambiguity neutrality and ambiguity aversion (for medium and high probabilities), because a cannot be simultaneously negative, zero and positive. By contrast, our model (Section 5) provides a parameter-free derivation of quantum probabilities and can explain the simultaneous occurrence in the same subject of ambiguity seeking (low probabilities), ambiguity neutrality and ambiguity aversion (medium and high probabilities). Its predictions are in good agreement with the empirical evidence in Dimmock et al. (2015).

Busemeyer and Bruza (2012, section 9.1.2) conclude "In short, quantum models of decision making can accommodate the Allais and Ellsberg paradoxes. But so can non-additive weighted utility models, and so these paradoxes do not point to any unique advantage for the quantum model". Note, however, that there is considerable arbitrariness in the choice of weights in weighted utility models. Hence, they introduce flexibility at the cost of lower predictive power. In our model, we replace weights with quantum probabilities which are parameter-free. Thus, our application of projective expected utility theory has a clear advantage over all other decision theories. Furthermore, projective expected utility can be extended to include reference dependence and loss aversion, to yield *projective prospect theory*, where decision weights are replaced with quantum probabilities. This would have a clear advantage over all the standard (non-quantum) versions of prospect theory.

Aerts, Sozzo, and Tapia (2014) formulate and study a quantum decision theory (QDT) model of the Ellsberg paradox. They consider one of the standard versions of the Ellsberg paradox. They consider a single urn with 30 red balls and 60 balls that are either yellow or black, the latter in unknown proportions. They use the heuristic of insufficient reason for the known distribution (red) but not for the unknown distribution (yellow or black). They prove that in their mode, the Ellsberg paradox reemerges if they use the heuristic of insufficient reason (or equal a' priori probabilities) for the unknown distribution. They, therefore, abandon this heuristic. They choose the ratio of yellow to black to fit the evidence from their subjects.

Although abandoning the heuristic of insufficient reason gives models tremendous flexibility, it also reduces their predictive power. In both classical (Kolmogorov) probability theory and quantum probability theory, any probabilities (provided they are non-negative and sum to 1) can be assigned to the elementary events. To make a theory predictive, some heuristic rule is needed to assign a' priori probabilities (we call this a heuristic because it does not follow from either classical or quantum probability theory). The heuristic commonly used is that of insufficient reason or equal a' priori probabilities.[5] This heuristic is crucial in deriving the Maxwell–Boltzmann distribution in classical statistical mechanics and the Bose–Einstein and Fermi–Dirac distributions in quantum statistical mechanics.[6] Furthermore, other theories can explain the Ellsberg paradox if we abandon *insufficient reason* (see Section 2.3). Thus, the explanation of Aerts et al. (2014) is not specifically quantum, although it is expressed in that language.

Khrennikov and Haven (2009) provide a general quantum-like framework for situations where Savage's sure-thing principle (Savage, 1954) is violated; one of these being the Ellsberg paradox. Their *quantum-like* or *contextual probabilistic* (*Växjö*) *model* is much more general than either the classical Kolmogorov model or the standard quantum model (see Haven & Khrennikov, 2013, and Khrennikov, 2010). On the other hand, our approach is located strictly within standard quantum theory. Furthermore, in their formulation, the Ellsberg paradox reemerges if one adopts (as we do) the heuristic of insufficient reason.[7]

We set up a simple quantum decision model of the Ellsberg paradox. We argue that our quantum decision model, in conjunction with the heuristic of insufficient reason, is in broad conformity with the evidence of Dimmock et al. (2015). In Table 1, the second column gives the means across 666 subjects of the observed matching probabilities for 0.1, 0.5 and 0.9. The third column gives the sample standard deviations. The fourth column gives the theoretical predictions of our model.

Our theoretical predictions of $m(0.5)$ and $m(0.9)$ are in excellent agreement with the average of observations. Our theoretical prediction of $m(0.1)$ is not statistically significantly different from the average of observed values.[8]

3 *Insufficient reason* or *equal a' priori probabilities* in now commonly referred to as *indifference*. However, indifference has a well established alternative meaning in economics. To avoid confusion, we shall use the older terminology.

4 The same reasoning can be repeated within any particular source in source dependent theory (see Section 3.5). So we have to take K and U as different sources.

5 To be sure, this heuristic is not without problems. See, for example, Gnedenko (1968), Sections 5 and 6, pp 37–52.

6 See Tolman (1938), Section 23, pp 59–62, for a good early discussion.

7 Khrennikov and Haven (2009), section 4.6, p 386.

8 For $m(0.1)$, $z = \frac{0.22-0.17105}{0.25} = 0.1958 < 1.96$. For such a large sample, the t-distribution is practically normal. Based on the normal test, the evidence does not reject the theoretical prediction $m(0.1) = 0.17105$ at the 5% level of significance.

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