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Monotonicity as a tool for differentiating between truth and optimality in the aggregation of rankings

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HIGHLIGHTS

- The aggregation of rankings, which is a common decision making problem in many fields of application, is addressed from a statistical point of view.
- The property of monotonicity of the profile, which serves as a tool for differentiating between truth an optimality in the aggregation of rankings, has been introduced.
- A noise model representing how people make mistakes, which results in both a measure of the most likely ranking and a statistical test for validating the real existence of such ranking, is proposed.
- The procedure has been illustrated with a real-life example concerning a decision making problem.

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ABSTRACT

The choice of the ranking that best captures the preferences of several voters on a set of candidates has been a matter of study for centuries. An interesting point of view on this problem is centred on the notion of monotonicity. In this paper, we deal with an aspect of monotonicity that has not been addressed before: if there is a true ranking on the set of candidates and every voter expresses a ranking on the set of candidates, then the number of times that each ranking is expressed should decrease when we move away from this true ranking in terms of pairwise discordances. In addition, we propose a probabilistic model that allows to formulate the choice of the best ranking as a maximum likelihood estimation problem. A test for the validity of this monotonicity assumption is also proposed.

1. Introduction

In 1785, Marie Jean Antoine Nicolas Caritat, mostly known as Condorcet (1785), followed the direction started by Rousseau in his remarkable work 'Du Contrat Social' (Rousseau, 1762) where he discusses about the 'general will': "When a law is proposed in the people's assembly, what is asked of them is not precisely whether they approve of the proposition or reject it, but whether it is in conformity with the general will [...]". In that way, Condorcet stated that it is not a compromise ranking what is searched, but an unknown truth that remains hidden due to the fact that "voters sometimes make mistakes in their judgements". In order to identify this unknown truth, Condorcet proposed a probabilistic model "of how the observed quantities depend probabilistically on the unobservable state of nature" for finding the ranking that is the "most likely to be best" (Young, 1988). In this same direction, Arrow (1963) stated a couple of centuries later the following: "[...] each individual has two orderings, one which governs him in his everyday actions, and one which would be relevant under some ideal conditions and which is in some sense truer than the first ordering. It is the latter which is considered relevant to social choice, and it is assumed that there is complete unanimity with regard to the truer individual ordering".

According to this philosophy where one supposes that there is a 'true' ranking on the set of candidates and voters attempt to

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identify this unknown truth, we propose in this paper to require the probabilities associated with the rankings to decrease in case we move away from the 'true' ranking. Intuitively, we advocate that, for a ranking $a \succ b \succ c$, the probability of a voter expressing the ranking $a \succ b \succ c$ should be greater than the probability of a voter expressing the ranking $a \succ c \succ b$; the latter probability should be at the same time greater than the probability of a voter expressing the ranking $c \succ a \succ b$; etc. We can interpret this natural requirement as some sort of monotonicity. Monotonicity is a common desired property in mathematical modelling, and its importance has been acknowledged in several disciplines, e.g. in machine learning (Ben-David, 1995; Cao-Van & De Baets, 2003; Lievens, De Baets, & Cao-Van, 2008) and fuzzy modelling (Perfilieva & De Baets, 2010; Stepnicka & De Baets, 2013; Van Broekhoven & De Baets, 2009). However, real-life data is often imperfect and does not fully comply with the monotonicity hypothesis. One option then is to (minimally) adjust the dataset restoring the monotonicity (Rademaker & De Baets, 2011; Rademaker, De Baets, & De Meyer, 2009, 2012). This is particularly important as, for instance, in machine learning, some algorithms cannot be trained with non-monotone datasets (Rademaker et al., 2009).

Returning to the field of social choice theory, several ranking rules centred on this monotonicity property (interpreting the monotonicity requirement in many different ways) have already been proposed. Rademaker and De Baets (2014) advocated that, for a ranking $a \succ b \succ c$, monotonicity implies that the number of voters preferring a to c should not be less than both the number of voters preferring a to b and the number of voters preferring b to c. Pérez-Fernández et al. (2016) formalized the former approach as the search for monotonicity of a natural representation of votes: the votrix.[1] Monotonicity of other representations of votes, such as the scorix (Pérez-Fernández, Rademaker, & De Baets, 2016) and the votex (Pérez-Fernández et al., 2016), has also been analysed in recent works, leading to the introduction of many different intuitive ranking rules. However, none of these ranking rules based on the search for monotonicity of a representation of votes has been shown to be a maximum likelihood estimator (for an appropriate probabilistic model) of the latent 'true' ranking on the set of candidates (Conitzer & Sandholm, 2005). In this paper, a ranking rule based on the natural property of monotonicity of the profile of rankings is proposed. Moreover, we introduce a probabilistic model introducing how people make mistakes that results in the identification of the most likely ranking, and, additionally, in a statistical test for validating the real existence of such ranking.

The rest of the paper is organized as follows. Section 2 is devoted to the search for an optimal ranking. In Section 3, the notion of Maximum Likelihood Monotone Estimator is introduced. This estimator is used subsequently to test the monotonicity assumption in Section 4. The methodology is illustrated on a real-life example in Section 5. Finally, we address some conclusions and open problems in Section 6.

2. Monotonicity of a profile of rankings

Social choice theory considers the problem where several voters express their preferences on a set $\mathscr{C}$ of k candidates. In the setting considered here, each of the r voters expresses his/her preferences on the set of candidates in the form of a ranking $\succ_j$ (the asymmetric part of a total order relation $\succeq_j$). The list $\mathscr{R} = (\succ_j)_{j=1}^r$ consisting of all the provided rankings is known as the profile of rankings given by the voters. A rule deciding which ranking is the winning ranking for a given profile of rankings is called a ranking rule.

In the following, we will provide some notations that will be used throughout this paper. The set of all possible rankings on $\mathscr{C}$ is denoted by $\mathcal{L}(\mathscr{C})$ and the set of all possible profiles of r rankings on $\mathscr{C}$ is denoted by $\mathcal{L}(\mathscr{C})^r$. Every ranking $\succ \in \mathcal{L}(\mathscr{C})$ is identified with a label $i \in \{1, \ldots, k!\}$ (for instance, determined by the lexicographical order on $\mathcal{L}(\mathscr{C})$). It is important to note that the labelling $i \in \{1, \ldots, k!\}$ of the rankings in $\mathcal{L}(\mathscr{C})$ does not coincide with the labelling $j \in \{1, \ldots, r\}$ of the rankings in the profile $\mathscr{R}$.

Any profile of rankings is determined by the number of times that each ranking is expressed.[2] We denote by $\mathbf{n}_{\mathscr{R}} \in \{0, 1, \ldots, r\}^{k!}$ the vector of absolute frequencies of $\mathscr{R}$, where $\mathbf{n}_{\mathscr{R}}(i)$ is the absolute frequency of the ith ranking in $\mathcal{L}(\mathscr{C})$, i.e., the number of voters that expressed the ith ranking in $\mathcal{L}(\mathscr{C})$ in the profile $\mathscr{R}$. Analogously, we denote by $\mathbf{f}_{\mathscr{R}} \in \{0, \frac{1}{r}, \ldots, 1\}^{k!}$ the vector of relative frequencies of $\mathscr{R}$, where $\mathbf{f}_{\mathscr{R}}(i)$ is the relative frequency of the ith ranking in $\mathcal{L}(\mathscr{C})$ in the profile $\mathscr{R}$. For any $i \in \{1, \ldots, k!\}$, it obviously holds that

$$r \cdot \mathbf{f}_{\mathscr{R}}(i) = \mathbf{n}_{\mathscr{R}}(i).$$

In addition, it holds that

$$\sum_{i=1}^{k!} \mathbf{n}_{\mathscr{R}}(i) = r \quad \text{and} \quad \sum_{i=1}^{k!} \mathbf{f}_{\mathscr{R}}(i) = 1.$$

The set of all possible vectors $\mathbf{f} \in \{0, \frac{1}{r}, \ldots, 1\}^{k!}$ that can be seen as the vector of relative frequencies of a profile $\mathscr{R}$ of r rankings on $\mathscr{C}$ is denoted by $\mathcal{R}_r(\mathscr{C})$. Analogously, the set of all possible unit vectors, i.e., all possible vectors $\mathbf{f} \in [0, 1]^{k!}$ such that $\sum_{i=1}^{k!} \mathbf{f}(i) = 1$, is denoted by $\mathcal{R}(\mathscr{C})$. Obviously, it holds that

$$\mathcal{R}_r(\mathscr{C}) \subseteq \mathcal{R}(\mathscr{C}).$$

Each ranking $\succ$ on $\mathscr{C}$ defines an order relation $\geqq$ on $\mathcal{L}(\mathscr{C})$ according to how far two rankings in $\mathcal{L}(\mathscr{C})$ are from $\succ$ in terms of reversals.[3] For any $\succ_i, \succ_j \in \mathcal{L}(\mathscr{C})$, the fact that $(\succ_i, \succ_j) \in \geqq$ is denoted by $\succ_i \geqq \succ_j$.

Proposition 1. *Let $\mathscr{C}$ be a set of k candidates and $\succ$ be a ranking on $\mathscr{C}$. The relation $\geqq$ defined as*

$$\geqq \; = \Big\{(\succ_i, \succ_j) \in \mathcal{L}(\mathscr{C})^2 \mid \big(\forall (a_{i_1}, a_{i_2}) \in \mathscr{C}^2\big) \\ \Big(\big((a_{i_1} \succ a_{i_2}) \wedge (a_{i_1} \succ_j a_{i_2})\big) \Rightarrow (a_{i_1} \succ_i a_{i_2})\Big)\Big\}$$

is an order relation on $\mathcal{L}(\mathscr{C})$.

Proof. We prove that $\geqq$ satisfies reflexivity, antisymmetry and transitivity.

Reflexivity: holds trivially.

Antisymmetry: for any $\succ_i, \succ_j \in \mathcal{L}(\mathscr{C})$, if $\succ_i \geqq \succ_j$ and $\succ_j \geqq \succ_i$, then it holds that:

$$\big(\forall (a_{i_1}, a_{i_2}) \in \mathscr{C}^2\big) \Big(\big((a_{i_1} \succ a_{i_2}) \wedge (a_{i_1} \succ_j a_{i_2})\big) \Rightarrow (a_{i_1} \succ_i a_{i_2})\Big),$$
$$\big(\forall (a_{i_1}, a_{i_2}) \in \mathscr{C}^2\big) \Big(\big((a_{i_1} \succ a_{i_2}) \wedge (a_{i_1} \succ_i a_{i_2})\big) \Rightarrow (a_{i_1} \succ_j a_{i_2})\Big).$$

Therefore, for any $a_{i_1}, a_{i_2} \in \mathscr{C}$ such that $a_{i_1} \succ a_{i_2}$, it holds that

$$(a_{i_1} \succ_i a_{i_2}) \Leftrightarrow (a_{i_1} \succ_j a_{i_2}).$$

[1] The fact that there exists a ranking w.r.t. which the votrix is monotone implies the property of strong stochastic transitivity (Rieskamp, Busemeyer, & Mellers, 2006). A weaker property, usually referred to as weak stochastic transitivity, has also called the attention of researchers (Regenwetter, Dana, & Davis-Stober, 2011).

[2] We consider here (anonymized) profiles of rankings (Pérez-Fernández, Rademaker, & De Baets, 2017), i.e., we do not take the order of the voters into account.

[3] A reversal is a switch of consecutive elements in a ranking. The minimum number of reversals needed for changing a given ranking into another one is measured by the Kendall distance function (Kendall, 1938).