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Some results on biordered structures, in particular distributed systems

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HIGHLIGHTS

- We collect necessary conditions for the continuous representability of biorders.
- The representability of biorders between preordered spaces is studied.
- Distributed systems and their representations are mathematically studied.
- We characterize the continuous representability of a biorder in the finite case.

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ABSTRACT

We study necessary conditions for the continuous representability of biorders. Furthermore, if the biorder is defined from a totally preordered space to another totally preordered space, then we obtain a characterization of the continuous representability of the biorder in case that the functions of the representation also represent the corresponding total preorders. This also supposes a mathematical formalization of the so called distributed system, common in computer sciences and causality. Finally, we solve the case of a biorder in which the quotient sets – with respect to the indifference of the traces – are finite and, in particular, the case in which the sets are finite.

1. Introduction and motivation

In this paper, we search for characterizations of the continuous representability of biorders. Besides, we use the concept of biorder in order to mathematically formalize the order structure of a distributed system.

The concept of a biorder was introduced by André Ducamp and Jean-Claude Falmagne in 1969 in Ducamp and Falmagne (1969), and studied in depth in 1984 by Jean-Paul Doignon, André Ducamp and Jean-Claude Falmagne in Doignon, Ducamp, and Falmagne (1984). It is defined as follows:

A biorder $\mathcal{R}$ from A to X is a binary relation, with $\mathcal{R} \subseteq A \times X$, satisfying that for every $a, b \in A$ and $x, y \in X$ $(a\mathcal{R}x) \wedge (b\mathcal{R}y)$ implies $(a\mathcal{R}y) \vee (b\mathcal{R}x)$.

Although we will work with biorders between two sets A and X, in the literature (Aleskerov, Bouyssou, & Monjardet, 2007; Doignon et al., 1984) the concept of biorder can also be found just as a Ferrers relation, that is, as a relation $\mathcal{R}$ on a single set X such that for any $x, y, z, t \in X$ it holds that $x\mathcal{R}y$ with $z\mathcal{R}t$ implies that $x\mathcal{R}t$ or $z\mathcal{R}y$. Anyway, notice that if $\mathcal{R}$ is a relation from A to X, then it is also a relation from A' to X', for any A' and X' such that $A \subseteq A'$ and $X \subseteq X'$. Therefore, $\mathcal{R}$ is also a relation on the single set $A \cup X$ (Doignon et al., 1984).

Thus, the concept of biorder includes the well-known concept of interval order (and consequently, the more restrictive one of semiorder) (Aleskerov et al., 2007; Bosi, Candeal, & Induráin, 2007; Bridges & Mehta, 1995; Fishburn, 1970; Luce, 1956; Scott & Suppes, 1958; Tversky, 1969), that actually is just an irreflexive biorder on a single set X (Aleskerov et al., 2007; Doignon et al., 1984).

The motivations to study *continuous* representability are various. We should observe that a numerical representation of a biorder translates the *qualitative* scale defined by the given biorder into a *quantitative* one, namely the real line with its usual order. Of course, it is much easier to compare numbers than to compare objects or elements belonging to an abstract set in a (just) qualitative way. Moreover, it is quite common to encounter scales in our word that vary in a continuous manner, so that the quantification that corresponds to the search for a suitable numerical representation should also be made in a way that preserves continuity.

For the particular case of an interval order several results on continuous representability appeared in the last years (Bosi, Campión, Candeal, & Induráin, 2007; Bosi, Estevan, Gutiérrez

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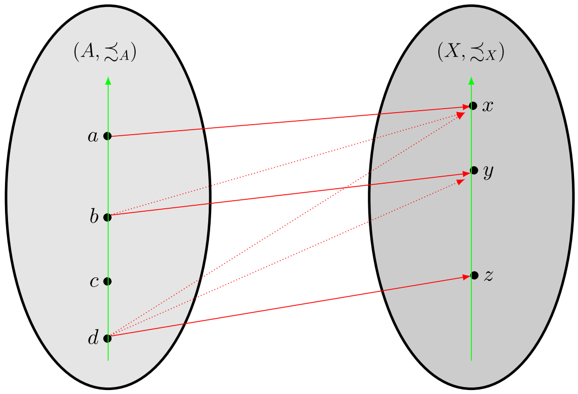

Fig. 1. Transitive reduction of a bordered pair of totally preordered sets. The green lines represent the total preorders and the red ones represent the biorder from A to X. Some of these lines are dashed in order to show the transitive reduction of the order structure. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

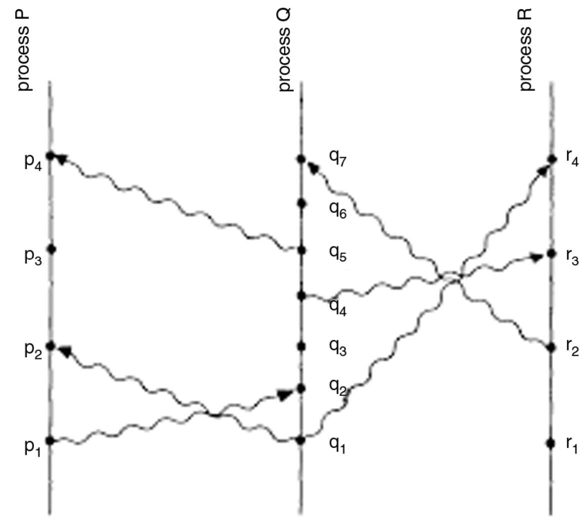

Fig. 2. Illustration of a distributed system, taken from the paper (Lamport, 1978) of Leslie Lamport.

García & Induráin, 2015; Estevan, Gutiérrez García, & Induráin, 2013a). However, there is a certain lack of results on continuous representability of biorders (though partial results on connected topological spaces are well-known Chateauneuf, 1987) and up-to-date *no general characterization of the continuous representability of a biorder is known yet.*

Here we introduce a new set of necessary conditions for the continuous representability of biorders. Special attention is paid to biorders on finite sets (and also to those biorders defined on sets whose quotients are finite with respect to the traces), where we prove that those necessary conditions for the continuous representability are also sufficient. We also focus on biorders defined between totally preordered sets (see Fig. 1), where a characterization of the continuous representability of the order structure is achieved too.

The study of these order structures, pairs of biordered and disjoint sets (that can be generalized to n disjoint sets $X_1, \ldots, X_n$ linked by $n(n-1)$ biorders defined from X_i to X_j, for any $i \neq j$), is also interesting dealing with distributed systems (Fidge, 1991; Lamport, 1978; Mattern, 0000; Raynal & Singhal, 1991). An *event* (illustrated by a point in Figs. 1 and 2) is a uniquely identified runtime instance of an atomic action of interest. It is an occurrence at a point in time, i.e., a happening at a cut of the timeline, which itself does not take any time. A *process* (illustrated by a vertical line in Figs. 1 and 2) is a sequence of a totally ordered events, i.e., for any event a and b in a process, either a comes before b or b comes before a. A *distributed system* consists of a collection of distinct processes which are spatially separated (so, disjoint from a mathematical point of view), and which communicate with one another by exchanging messages: this communication (illustrated by red arrows in Fig. 1 and wavy arrows in Fig. 2) implies an order between events of different processes. It is assumed that sending or receiving a message is an event.

Since each process consists of a sequence of events, each process is a totally ordered set (we will generalize it to totally preordered sets), and the communication through messages between the processes can be mathematically formalized by means of biorders.

Moreover, this communication between processes defines a causal relation known as 'causal precedence' or 'happened before' relation (Fidge, 1991; Lamport, 1978; Mattern, 0000; Raynal & Singhal, 1991) (common in causality too Panangaden, 2014, but now related to the theory of relativity, see also Busemann, 1967, Kronheimer & Penrose, 1966), which is a strict partial order on the elements of the sets. This causal relation is a powerful concept for reasoning, analysing and drawing inferences about distributed computation (Raynal & Singhal, 1991). The *causal precedence* (denoted by $\rightarrow$) on the set of events of a system is defined by Leslie Lamport (Lamport, 1978) as the smallest relation satisfying the following three conditions:

1. If a and b are events in the same process, and a comes before b, then $a \rightarrow b$.
2. If a is the sending of a message by one process and b is the receipt of the same message by another process, then $a \rightarrow b$.
3. If $a \rightarrow b$ and $b \rightarrow c$ then $a \rightarrow c$.

This definition was introduced by Leslie Lamport in 1978 (see Lamport, 1978) and it has been used until nowadays. In the present paper, we shall introduce a mathematical definition of the concept through orderings.

Graphically, $\alpha \rightarrow \beta$ means that one can follow a 'path of causality' from event α to event β in the diagram, moving in the direction of the arrows (see Fig. 1 or Fig. 2).

Two distinct events a and b are concurrent if $a \nrightarrow b$ and $b \nrightarrow a$. Assume $a \nrightarrow a$ for any event a, so $\rightarrow$ is an irreflexive partial ordering on the set of all events in the system. The ordering is only partial because events can be concurrent in which case it is not known which event happened first. Two events a and b are concurrent if they in no way can causally affect each other, that is, there is no causality between them.

Given a preorder $\precsim$ on X, a real function $u: X \rightarrow \mathbb{R}$ is said to be *isotonic* or *increasing* if for every $x, y \in X$ the implication $x \precsim y \Rightarrow u(x) \leq u(y)$ holds true. In the case of a total preorder $\precsim$ on X, it is said to be *representable* if there is a real-valued function $u: X \rightarrow \mathbb{R}$ that is *strictly isotonic* or *strictly increasing* (also known as *order-preserving*), so that, for every $x, y \in X$, it holds that $x \precsim y \iff u(x) \leq u(y)$. The map u is said to be an *order-monomorphism* (also known as a *utility function* for $\precsim$).

A (not necessarily total) preorder $\precsim$ on a set X is said to have a *multi-utility representation* if there exists a family $\mathcal{U}$ of isotonic real functions such that for all points $x, y \in X$ the equivalence $x \precsim y \Leftrightarrow \forall u \in \mathcal{U}\, u(x) \leq u(y)$ holds. This kind of representation, whose main feature is to fully characterize the preorder, was first introduced by Levin (1985) (see also Levin, 2001), who called *functionally closed* a preorder admitting amulti-utility representation. However, the

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