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Possibilities determine the combinatorial structure of probability polytopes

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ABSTRACT

We study the set of no-signalling empirical models on a measurement scenario, and show that the combinatorial structure of the no-signalling polytope is completely determined by the possibilistic information given by the support of the models. This is a special case of a general result which applies to all polytopes presented in a standard form, given by linear equations together with non-negativity constraints on the variables.

1. Introduction

This paper is concerned with the combinatorial structure of the polytopes of probability models which are widely studied in quantum information and foundations. Much current study focuses on the **no-signalling polytope** (Pironio, Bancal, & Scarani, 2011; Popescu, 2014), comprising those probability models whose correlations are consistent with the constraints imposed by relativity, and on characterizing the set of quantum correlations (Navascués, Pironio, & Acín, 2008), which is contained within this polytope.

As is well known, quantum correlations exceed those which can be achieved using classical, "local realistic" models. While Bell's original proof of this fact (Bell, 1964) used the detailed probabilistic structure of the models arising from quantum mechanics to show that they violated the Bell inequalities which hold for classical models, subsequent proofs by Greenberger, Horne, Shimony, and Zeilinger (1990); Greenberger, Horne, and Zeilinger (1989), Hardy (1992, 1993), and others (Cabello, 2001; Mermin, 1990; Zimba & Penrose, 1993), were inequality-free, and even probability-free. In Abramsky and Brandenburger (2011), a hierarchy of forms of non-locality, or more generally contextuality, was defined. The higher levels, of **logical contextuality** (generalizing Hardy arguments), **strong contextuality** (generalizing GHZ arguments), and **All-versus-Nothing contextuality** (Abramsky, Barbosa, Kishida, Lal, & Mansfield, 2015), use only the **possibilistic** information from the probability models. That is, they need only the information about which events are possible (probability greater than zero) or impossible (probability zero). In other words, they only refer to the **supports** of the probability distributions. Passing from probability models to their supports (the "possibilistic collapse") evidently loses a great deal of information. In this paper, we show that nevertheless, the **combinatorial structure** of the no-signalling polytope is completely captured by the possibilistically collapsed models, thus confirming that much structural information can in fact be gained from these apparently much simpler models.

In more precise terms, the **combinatorial type** of a polytope is given by its **face lattice** (Ziegler, 1995), the set of faces of the polytope, ordered by inclusion. These face lattices have a rich structure, and have been extensively studied in combinatorics.

Our main result can be stated as follows.

Theorem 1.1. *Fix a "measurement scenario", specifying a set of variables which can be observed or measured, the possible outcomes of these measurements, and which variables are compatible and can be measured together. We can then define a polytope $\mathcal{N}$ of no-signalling probability models over this scenario. Call the face lattice of this polytope $\mathcal{F}$. Now let $\mathcal{S}$ be the set of supports or possibilistic collapses of the models in $\mathcal{N}$. $\mathcal{S}$ is naturally ordered by context-wise inclusion of supports. Then there is an order-isomorphism $\mathcal{F} \cong \mathcal{S}$.*

Thus the combinatorial type of the polytope $\mathcal{N}$ is completely determined by its possibilistic collapse.

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This result has a number of interesting corollaries. For example:

- All the models in the relative interior of each face $F \in \mathcal{F}$ have the same support.
- The vertices of $\mathcal{N}$ are exactly the probability models with minimal support in $\mathcal{N}$. Moreover, there is only one probability model in $\mathcal{N}$ for each such minimal support.
- Thus the extremal probability models are completely determined by their supports.
- The vertices of $\mathcal{N}$ can be written as the disjoint union of the non-contextual, deterministic models – the vertices of the polytope of classical models – and the strongly contextual models with minimal support.

An empirical model has full support iff it is in the relative interior of $\mathcal{N}$. Consequently, any logically contextual model must lie in a proper face of the polytope.

In fact, this result applies to a much wider class of polytopes. Note that the no-signalling polytope, for any given measurement scenario, is defined by the following types of constraint:

- Non-negativity;
- Linear equations: namely normalization, and the no-signalling conditions.

In geometric terms, this says that $\mathcal{N} = \mathcal{H}_{\geq \mathbf{0}} \cap \mathsf{Aff}(\mathcal{N})$, where $\mathsf{Aff}(\mathcal{N})$ is the affine subspace generated by $\mathcal{N}$, and $\mathcal{H}_{\geq \mathbf{0}}$ is the non-negative orthant, *i.e.* the set of all vectors $\mathbf{v}$ with $\mathbf{v} \geq \mathbf{0}$.

It is a standard result that every linear program can be put in a "standard form" of this kind (Matousek & Gärtner, 2007), so that its associated polytope of constraints is of the type we are considering.

Now our theorem in fact applies at this level of generality.

Theorem 1.2 (*General Version*). *Let P be a polytope such that $P = \mathcal{H}_{\geq \mathbf{0}} \cap \mathsf{Aff}(P)$. Let $\mathcal{F}(P)$ be the face lattice of P. Let $\mathcal{S}(P)$ be the set of "supports" of points in P, i.e. 0/1-vectors where each positive component of $\mathbf{v} \in P$ is mapped to 1, while each zero component is left fixed. $\mathcal{S}(P)$ is naturally ordered componentwise, with $0 < 1$. Then there is an order-isomorphism $\mathcal{F}(P) \cong \mathcal{S}(P)$.*

The structure of the remainder of the paper is as follows. In Section 2, we provide background on measurement scenarios and empirical models. In Section 3 we review some basic notions on partial orders and lattices, and in Section 4, we review some standard material on polytopes. We prove our main result in Section 5, and apply it to probability polytopes, in particular no-signalling polytopes, in Section 6.

2. Measurement scenarios and empirical models

We shall give a brief introduction to the basic notions of measurement scenarios and empirical models, as developed in the sheaf-theoretic approach to contextuality and non-locality. For further discussion, motivation and technical details, see Abramsky and Brandenburger (2011) and Abramsky et al. (2015). An extended introduction and overview is given in Abramsky (2015).

A **measurement scenario** is a triple $(X, \mathcal{M}, O)$ where:

- X is a set of variables which can be measured, observed or evaluated.
- $\mathcal{M}$ is a family of sets of variables, those which can be measured together. These form the **contexts**.
- O is a set of possible outcomes or values for the variables.

In this paper, we shall only consider **finite** measurement scenarios, where the sets X, O, and hence also $\mathcal{M}$ are finite. This allows us to avoid measure-theoretic technicalities, while capturing the primary objects of interest in quantum information and foundations.

Given a measurement scenario $(X, \mathcal{M}, O)$, an **empirical model** for this scenario is a family $\{e_C\}_{C \in \mathcal{M}}$ of probability distributions

$$e_C \in \mathsf{Prob}(O^C), \quad C \in \mathcal{M}.$$

Here we write $\mathsf{Prob}(X)$ for the set of probability distributions on a finite set X. Such distributions are simply represented by functions $d : X \to [0, 1]$ which are normalized:

$$\sum_{x \in X} d(x) = 1.$$

The probability of an event $S \subseteq X$ is then given by

$$d(S) = \sum_{x \in S} d(x).$$

The set O^C is the set of all assignments $s : C \to O$ of outcomes to the variables in the context C. Such an assignment represents a **joint outcome** of measuring all the variables in the context.

Given $e_C \in \mathsf{Prob}(O^C)$, and a subset $U \subseteq C$, we have the operation of restricting e_C to U by **marginalization**:

$$e_C|_U \in \mathsf{Prob}(O^U),$$

defined by:

$$e_C|_U(s) = \sum_{t \in O^C,\ t|_U = s} e_C(t).$$

Here $t|_U$ is the restriction of the assignment t to the variables in U.

We say that an empirical model $\{e_C\}_{C \in \mathcal{M}}$ is **no-signalling** if for all $C, C' \in \mathcal{M}$:

$$e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}.$$

Thus an empirical model is no-signalling if the marginals from any pair of contexts to their overlap agree. This corresponds to a general form of an important physical principle. Suppose that $C = \{a, b\}$, and $C' = \{a, b'\}$, where a is a variable measured by an agent Alice, while b and b' are variables measured by Bob, who may be spacelike separated from Alice. Then under relativistic constraints, Bob's choice of measurement – b or b' – should not be able to affect the distribution Alice observes on the outcomes from her measurement of a. This is captured by saying that the distribution on $\{a\} = \{a, b\} \cap \{a, b'\}$ is the same whether we marginalize from the distribution e_C, or the distribution $e_{C'}$. The general form of this constraint is shown to be satisfied by the empirical models arising from quantum mechanics in Abramsky and Brandenburger (2011).

Example. Consider the following table:

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
a	b	0	1/2	1/2	0
a'	b	3/8	1/8	1/8	3/8
a	b'	3/8	1/8	1/8	3/8
a'	b'	3/8	1/8	1/8	3/8

This represents a situation where Alice and Bob can each choose measurement settings and observe outcomes. Alice can choose the settings a or a', while, independently, Bob can choose b or b'. The total set of variables is $X = \{a, a', b, b'\}$. The **measurement contexts** are

$$\mathcal{M} = \{\{a, b\},\ \{a', b\},\ \{a, b'\},\ \{a', b'\}\}.$$

Each measurement has possible outcomes in $O = \{0, 1\}$. The matrix entry at row (a', b) and column $(0, 1)$ corresponds to the **event**

$$\{a' \mapsto 0,\ b \mapsto 1\}.$$

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