Journal of Mathematical Psychology ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

An introduction to lattice based probability theories☆

Louis Narens

Department of Cognitive Sciences, University of California, Irvine, United States

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Foundations of probability
Lattice theory
Intuitionistic logic
Quantum probability theory

ABSTRACT

The focus of this article is on probability theories based on non-boolean event spaces that have potential for scientific and philosophical applications. Since the early 1930s, such an alternative probability theory has been systematically used to provide a mathematical foundation for quantum physics. This article investigates in a general way feasible alternatives to boolean event algebras as bases for probability theories. Such alternatives permit new kinds of logical relationships among events that are not expressible in boolean event algebras. In psychology, these relationships allow for kinds of modeling methods that help account for various puzzling phenomena found throughout psychology, particularly in cognitive decision theory.

The alternative probability theories presented are investigated through algebraic characterizations of their event spaces and probability functions. This is done using results from a well-developed area of mathematics known as "lattice theory". This article describes the basics of lattice theory and how the interpretation of a few of its results indicates that generalizations of boolean event spaces are severely limited for useful application in scientific modeling. The overall conclusion is that two types of event spaces appear especially promising for generalization: One that captures key concepts of the logic inherent in quantum mechanics, "quantum logic", and another that captures key concepts inherent in a generalization of classical logic that is used in the foundations of mathematics, "intuitionistic logic". Both of these types of logical structures are useful for constructing unorthodox models of troublesome findings in behavioral economics. One theme of this article is the use of intuitionistic logic to model probabilistic judgments and decision making.

1. Introduction

Probability theory began in 1654 with the mathematicians Blaise Pascal and Pierre de Fermat providing advice to the gambler Chevalier de Mere. Inherent in gambling games is the premise that the outcome of the gamble will be completely determined at some future time to the satisfaction of the participants: that is, it will be determined that event on which the gamble is based has occurred or it will be determined that its complement has occurred. This imposes a condition called "the Law of the Excluded Middle" on the logical structure of the space of gambling events. Boole (1854) formulated a logical structure for gambling events that today is called "a boolean algebra of events". One of its conditions is the Law of the Excluded Middle. There are kinds of gambling situations for which the Law of the Excluded Middle fails.

One example is betting on the truths of arithmetical statements. It is well-known that there is no universal method of determining which statements of an explicit formulation of arithmetic are true and which are false, thus producing a failure of the Law of the Excluded Middle (from the epistemic perspective of the betters, since they do not expect all of the statements they are betting on to have determinate truth values). "Intuitionistic logic" is a non-boolean logic formulated by Heyting (1930) to deal with this kind of "incompleteness". It produces a more general event algebra than a boolean algebra of events, and its probabilistic use requires a more generalized form of a probability function than the one traditionally employed for a boolean algebra of events.

Another kind of probabilistic event algebra appeared in the early formulation of quantum mechanics (Dirac, 1930; von Neumann, 1932). It also was a generalization of a boolean algebra of events, and it required a corresponding generalization of the notion of probability function. The quantum version allows for events A and B such that the smallest event including them, denoted by $A \Cup B$, is different from their set-theoretic union, $A \cup B$.

Except for quantum mechanics, probability theories based on boolean algebra of events have come to dominate science and

☆ The research for this article was supported by grant FA9550-13-1-0012 from AFOSR and grant SMA-1416907 from NSF.

E-mail address: lnarens@uci.edu.

http://dx.doi.org/10.1016/j.jmp.2016.04.013
0022-2496/

Please cite this article in press as: Narens, L., An introduction to lattice based probability theories. Journal of Mathematical Psychology (2016), http://dx.doi.org/10.1016/j.jmp.2016.04.013

philosophy. This article investigates alternatives to the boolean theories. The development is done in a manner designed to provide insight into why the quantum and intuitionistic generalizations or their close algebraic relatives appear to be the few options for useful generalizations of "event space" and "probability function" for possible scientific inquiry and application.

This article is designed to be an introduction to lattice theory and its application to the mathematical and philosophical foundations of probability. Much of the article's material is based on the development of Narens (2015) and the proofs of many of the presented theorems can be found in that work.

2. Boolean probability theory

Formally, a **boolean algebra of events** is a set $\mathcal{X}$ of subsets of a nonempty set X such that

- X and the empty set, $\varnothing$, are in $\mathcal{X}$,
- $X \neq \varnothing$, and
- if A and B are in $\mathcal{X}$, then $A \cup B$, $A \cap B$, and $-A$ are in $\mathcal{X}$, where $\cup$, $\cap$, and $-$ are, respectively, set-theoretic union, intersection, and complementation with respect to X.

Elements of $\mathcal{X}$ are called **events**, X is called the **sure event,** and the empty set, $\varnothing$, is called the **impossible** or **null event**. Elements of X are interpreted as **possible states** of the world, or just **states** for short. Among the possible states of the world is the **true state**, which is generally unknown. Associated with $\mathcal{X}$ is a **boolean probability function** $\mathbb{P}$. For events A in $\mathcal{X}$, $\mathbb{P}(A)$ is interpreted as the degree of belief that the true state of the world is in A. $\mathbb{P}$ is assumed to have the following properties for all A and B in $\mathcal{X}$:

- $\mathbb{P}(X) = 1$, $\mathbb{P}(\varnothing) = 0$, and $0 \subseteq \mathbb{P}(A) \subseteq 1$.
- **Boolean finite additivity**: If $A \cap B = \varnothing$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

For some applications, a stronger condition than boolean finite additivity is needed:

- **boolean σ-additivity**: For each countable set of pairwise disjoint events $A_1, A_2, \ldots$ in $\mathcal{X}$ such that $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{X}$,

$$\mathbb{P}\left(\bigcup_{i}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

For the purposes of this article, only the more general case of boolean finite additivity is considered.

There are a number of ways to generalize boolean probability theory. In the literature, particularly the psychological literature, the generalizations usually take the form of assuming a non-finitely additive probability function on a boolean algebra of events. This article focuses on a different strategy for generalization: generalizing both the underlying boolean algebra of events and its probability function so that the probability function retains a useful form of finite additivity. This second kind of generalization is useful for understanding the impact of context on judgment and measurement—a ubiquitous and important issue throughout science.

3. Event lattices

In a boolean algebra of events $\mathcal{X}$, $\cup$ and $\cap$ can be characterized in terms of the set-theoretic subset relation, $\subseteq$, as follows: For all A and B in $\mathcal{X}$,

- $A \cup B =$ the $\subseteq$-smallest set C in $\mathcal{X}$ such that $A \subseteq C$ and $B \subseteq C$, and
- $A \cap B =$ the $\subseteq$-largest set D in $\mathcal{X}$ such that $D \subseteq A$ and $D \subseteq B$.

The above definition of $A \cup B$ says that C is the $\subseteq$-least upper bound in $\mathcal{X}$ of A and B and that $A \cap B$ is the $\subseteq$-greatest lower bound of A and B in $\mathcal{X}$. The suggests the following generalization of $\cup$ and $\cap$ and "boolean algebra of events":

Definition 1. $\mathfrak{X} = \langle \mathcal{X}, \Cup, \Cap, X, \varnothing \rangle$ is said to be a **lattice algebra of events** if and only if $X \neq \varnothing$, $\mathcal{X}$ is a set of subsets of X, and $\Cup$ and $\Cap$ are binary operations on $\mathcal{X}$ such that the following statements hold for all A and B in $\mathcal{X}$:

- X and $\varnothing$ are in $\mathcal{X}$ and $X \neq \varnothing$.
- $A \Cup B$ is the $\subseteq$-smallest element of $\mathcal{X}$ such that $A \subseteq A \Cup B$ and $B \subseteq A \Cup B$.
- $A \Cap B$ is the $\subseteq$-largest element of $\mathcal{X}$ such that $A \Cap B \subseteq A$ and $A \Cap B \subseteq B$.

As previously, for lattice algebra of events defined above, elements of $\mathcal{X}$ are called "events", and previous event concepts, e.g., "true state of the world", "sure event", etc., have analogous definitions. □

The *algebra* inherent in boolean algebra of events is captured by the following definition and theorem.

Definition 2. Let $\mathfrak{X} = \langle \mathcal{X}, \Cup, \Cap, X, \varnothing \rangle$ be a lattice algebra of events.

$\dot{-}$ is said to be a **complementation operation** on $\mathfrak{X}$ if and only if $\dot{-}$ is a unary operation on $\mathcal{X}$ such that for all A in $\mathcal{X}$,

$$A \Cup \dot{-}A = X \text{ and } A \Cap \dot{-}A = \varnothing.$$

$\mathfrak{X}$ is said to be **complemented** if and only if there exists a complementation operation on $\mathfrak{X}$.

$\mathfrak{X}$ is said to be **distributive** if and only if for all A, B, and C in $\mathcal{X}$,

$$A \Cap (B \Cup C) = (A \Cap B) \Cup (A \Cap C).$$

A lattice algebra of events is said to be **boolean** if and only if it is complemented and distributive. A description of a boolean lattice algebra of events usually has the form $\langle \mathcal{X}, \Cup, \Cap, \dot{-}, X, \varnothing \rangle$, where $\dot{-}$ is a complementation operation on $\mathcal{X}$. A lattice algebra of events of the form $\langle \mathcal{Y}, \cup, \cap, Y, \varnothing \rangle$ is called a **distributive algebra of events**. Note that a boolean algebra of events $\langle \mathcal{X}, \cup, \cap, X, -, \varnothing \rangle$ is the special case of a boolean lattice algebra of events, $\langle \mathcal{X}, \Cup, \Cap, \dot{-}, X, \varnothing \rangle$, where $\Cup = \cup$, $\Cap = \cap$, and $\dot{-} = -$. □

There are two notions of isomorphism that can be productively defined to hold between two lattice algebras of events $\mathfrak{X} = \langle \mathcal{X}, \Cup, \Cap, X, \varnothing \rangle$ and $\mathfrak{Y} = \langle \mathcal{Y}, \Cup', \Cap', Y, \varnothing \rangle$. First, $\mathfrak{X}$ and $\mathfrak{Y}$ are said to be **lattice isomorphic** if and only if there exists a one-to-one function α *from* $\mathcal{X}$ *onto* $\mathcal{Y}$ such that for all A and B in $\mathcal{X}$,

$$\alpha(X) = Y, \quad \alpha(\varnothing) = \varnothing, \quad \alpha(A \Cup B) = \alpha(A) \Cup' \alpha(B), \quad \text{and}$$
$$\alpha(A \Cap B) = \alpha(A) \Cap' \alpha(B).$$

The second notion of isomorphism is a structure preserving function β *from* $\mathcal{X}$ *onto* $\mathcal{Y}$: For each Z in $\mathcal{X}$, let

$$\beta(Z) = \{\beta(x) \mid x \in Z\}.$$

Then β is said to be an **event isomorphism** from $\mathfrak{X}$ onto $\mathfrak{Y}$ if and only if β is a one-to-one function *from* X *onto* Y such that for all A and B in $\mathcal{X}$,

$$\beta(A \Cup B) = \beta(A) \Cup' \beta(B), \quad \text{and} \quad \beta(A \Cap B) = \beta(A) \Cap' \beta(B).$$

All event isomorphisms are lattice isomorphisms. An example is given below of a lattice isomorphism from a boolean lattice algebra of events onto a boolean algebra of events that is not an event isomorphism.

Theorem 1 (Stone, 1936, 1937)**.** *Let $\mathfrak{X} = \langle \mathcal{X}, \Cup, \Cap, X, \varnothing \rangle$ be a distributive lattice algebra and $\mathfrak{Y} = \langle \mathcal{Y}, \Cup_1, \Cap_1, \dot{-}, Y, \varnothing \rangle$ be a boolean lattice algebra of events. Then the following two statements hold:*