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Quantum cognition and decision theories: A tutorial

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HIGHLIGHTS

- We present a tutorial on quantum models of cognition and decision aimed at those with little or no prior experience of such models.
- We focus on the question of how to build quantum models in practice.
- We give examples from the study of order effect, including a new derivation of the QQ Equality.

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ABSTRACT

Models of cognition and decision making based on quantum theory have been the subject of much interest recently. Quantum theory provides an alternative probabilistic framework for modelling decision making compared with classical probability theory, and has been successfully used to address behaviour considered paradoxical or irrational from a classical point of view.

The purpose of this tutorial is to give an introduction to quantum models, with a particular emphasis on how to build these models in practice. Examples are provided by the study of order effects on judgements, and we will show how order effects arise from the structure of the theory. In particular, we show how to derive the recent discovery of a particular constraint on order effects implied by quantum models, called the Quantum Question (QQ) Equality, which does not appear to be derivable from alternative accounts, and which has been experimentally verified to high precision. However the general theory and methods of model construction we will describe are applicable to any quantum cognitive model. Our hope is that this tutorial will give researchers the confidence to construct simple quantum models of their own, particularly with a view to testing these against existing cognitive theories.

1. Introduction

Models of decision making based on the mathematics of quantum theory have attracted a large amount of interest recently (Aerts, 2009; Busemeyer & Bruza, 2014; Khrennikov, 2010; Mogiliansky, Zamir, & Zwirn, 2009; Pothos & Busemeyer, 2013; Wang, Busemeyer, Atmanspacher, & Pothos, 2013; Yukalov & Sornette, 2011). These models have arisen in part as a response to the empirical challenges faced by 'rational' decision-making models, such as those based on Bayesian probability theory. (Such examples are mostly associated with the famous Tversky–Kahneman research tradition. See e.g. Chater, Tenenbaum, & Yuille, 2006; Tversky & Kahneman, 1974.) These quantum models posit that, at least in some circumstances, human behaviour does not align well with classical probability theory or expected utility maximisation. However unlike, for example, the fast and frugal heuristics programme (see, e.g. Gigerenzer, Hertwig, & Pachur, 2011), quantum cognition aims not to do away with the idea of a formal structure underlying decision-making, but simply to replace the structure of classical probability theory with an alternative theory of probabilities. This new probability theory has features, such as context effects, interference effects and constructive judgements, which align well with psychological intuition about human decision-making. Initial research involving quantum models tended to focus mainly on explaining results previously seen as paradoxical from the point of view of classical probability theory, and there have been a number of successes in this area (Aerts, Gabora, & Sozzo, 2013; Blutner, Pothos, & Bruza, 2013; Bruza, Kitto, Ramm, & Sitbon, 2015; Pothos & Busemeyer, 2009, 2013; Trueblood & Busemeyer, 2011; Wang et al., 2013; White, Pothos, & Busemeyer, 2014). More

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recently, the focus has switched to some extent to testing new predictions arising from quantum models, and designing better tests of quantum vs. classical decision theories (Atmanspacher & Filk, 2010; Yearsley & Pothos, 2014, in preparation).

One key success of quantum models of cognition has been the treatment of question order effects (Moore, 2002). 'Order effects' here describes a phenomenon where, for example, given two particular questions, each with a number of possible responses, the expected distribution of responses to a particular question depends on whether it was asked first or second in the series. In other words, asking a prior question can influence the outcome of a subsequent one. We will explain in more detail below exactly how to characterise this effect.

As we shall see, order effects arise naturally in quantum theory, and thus they can be accounted for by quantum cognitive models. However what is more remarkable is that quantum theory also predicts particular constraints on the probabilities that can be generated by these models, most notably in the form of the Quantum Question (QQ) Equality (Wang & Busemeyer, 2013). These constraints seem to be extremely well satisfied in the data from real world experiments and surveys (Wang, Solloway, Shiffrin, & Busemeyer, 2014). Thus as well as being a natural application of quantum theory, question order effects also represent a striking empirical confirmation of the idea of using quantum theory to model decisions.

Although the mathematical machinery of quantum theory is not inherently more complex than that required by many other cognitive models, essentially linear algebra and a small amount of calculus, it is rather unfamiliar to most cognitive scientists. Our aim in this tutorial paper is therefore to introduce readers to the ideas and machinery of basic quantum theory, such that after working their way through this tutorial readers will feel more confident making use of quantum models in their research.

As well as existing cognitive scientists, we hope this tutorial may find a secondary audience in those researchers who already have a background in quantum theory gained from studying the physical sciences, who are interested in the application of these ideas in social science.[1] To help these readers we have structured our discussion of the basic formalism of quantum theory in a way which should feel familiar to anyone who first encountered it in the context of the physical sciences (see for example Isham, 1995 or the notes by Plenio, 2002 available online). Hopefully this should enable those already familiar with quantum theory to quickly grasp how to apply their existing knowledge to the construction of cognitive models.

The material we will cover in this tutorial is essential background to any application of quantum theory in judgement and decision making. We will pay particular attention to two important but sometimes overlooked issues; first how exactly does one choose a particular framework of Hilbert space, basis vectors, initial state etc. to suit the problem at hand, and what do these choices mean? Second, how are the various calculations actually carried out? Grasping both of these issues is essential for any student of the field, and we hope this tutorial will help researchers bridge the gap between reading about quantum models and actually constructing them for themselves.

There are a number of things we will not cover in this tutorial, which may be worth stating now. First, although we will mention it, we will not cover the dynamics of quantum systems in any detailed way; this is mainly a tutorial on quantum statics. Quantum dynamics are relatively simple to grasp once one understands the material in this tutorial. Second, we will not touch upon 'entanglement' or issues around quantum information. Finally, some advanced topics, such as CP-Maps and the full theory of POVMs will not be covered, as they are best learnt about once one is familiar with the basics. They will be covered in a subsequent tutorial (Yearsley, in preparation).

We will assume the reader has a good familiarity with linear algebra in the usual form of vectors, matrices etc., but for reference, and to set notation, we give a brief summary of some important ideas in the Appendix.

The rest of this tutorial is structured as follows; in Section 2 we introduce the basic elements of quantum cognition. In Section 3 we then expand upon some points, with the aim of guiding readers through the process of constructing a quantum model in practice rather than in theory. In Section 4 we give a brief introduction to order effects in quantum theory, and in Section 5 we expand on this to include a derivation of the QQ Equality. In Section 6 we give a brief introduction to POVMs, which can be used to represent noisy or imperfect measurements, and in Section 7 we apply these in the setting of order effects, our goal being to see to what extent the QQ Equality generalises to the case of more realistic noisy measurements. In Section 8 we briefly talk about another application of quantum theory to modelling similarity judgements. We summarise in Section 9. A number of mathematical details are contained in the Appendix.

2. The basics of quantum cognition

The aim of this section is to present the basic formalism of quantum cognition, including information about the state, the dynamics, and the description of measurements. Our goal here is to give a reasonably concise account of the essentials; in the next section we will return to each element in turn and ask in more detail what it means and how it may be specified for a particular model. We hope this format will make it easy for readers to grasp the essential structure of quantum models. All of the material in this section is standard, and we will not give references for individual results/definitions. For a more compete account see Isham (1995) or for an alternative description with a more cognitive focus see Busemeyer and Bruza (2014).

2.1. What is quantum cognition?

Quantum cognition is a framework for constructing cognitive models based on the mathematics of quantum probability theory, which is itself a mathematical framework for assigning probabilities to events, much like classical probability theory (for a full account see Busemeyer & Bruza, 2014). For a given event, usually thought of as the outcome of some judgement process, and specification of the decision maker by means of a cognitive state, quantum cognition gives a real number between 0 and 1 which is to be interpreted as the probability that the decision maker will make that particular choice. Quantum cognition also includes information about the set of possible dynamics, state transformations and measurements that can be performed on a system, although to a large extent this follows directly from the basic probabilistic structure.

In its most conservative form, quantum cognition is simply an algorithm for computing probabilities, without any claim to reflect the underlying way decisions are made in the brain. In this way of thinking, the success or otherwise of the approach is to be judged purely by the empirical success of its predictions. However steps are being taken towards viewing quantum cognitive theories as *process* models, that do reflect in some way the process of arriving at a given decision (Kvam, Pleskac, YU, & Busemeyer, 2015). The attraction of quantum models in this case stems in part from

[1] Indeed one of the present authors (JMY) has such a background.

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