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# Multi-stage sequential sampling models with finite or infinite time horizon and variable boundaries

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## HIGHLIGHTS

- The multi-stage decision model describes unfolding cognitive processes as piecewise diffusion processes.
- The model includes finite and infinite time horizons.
- The model is extended to account for non-constant decision boundaries.
- A Markov chain approach is implemented to account for nonstationary and nonlinear properties.

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ABSTRACT

The multi-stage decision model, aka multiattribute attention switching model, assumes a separate sampling process for each attribute and switching attention from one attribute to the next in a sequential fashion during one trial. Here the model is extended to finite and infinite time horizons and to non-constant boundaries. For a finite time horizon the model predicts a probability of not deciding within the available time. Two different families of non-constant boundaries are implemented, one with a nonlinear decrease, one with a constant boundary at the beginning and a linear decrease towards the deadline. Furthermore, it is shown how the stochastic process underlying each attribute of the multi-stage model (Wiener or Ornstein–Uhlenbeck process) can be discretized by a birth–death chain to implement all the relevant model features and how to provide speeded calculations. Several numerical examples are provided demonstrating the effect of the order of attribute processing (order schedule) and boundary properties. It is shown that, regardless of the time horizon or the non-constant boundaries, the order schedule is the determinant to predict a consistent choice probability/choice response time pattern including preference reversals and fast errors.



## 1. Introduction

Sequential sampling models of decision making have become the dominant approach to modeling decision processes in cognitive science. These models are designed to account for all three of the most basic dependent variables of cognitive psychology, which include choice, decision time, and confidence. Their application includes a variety of psychological tasks, from basic perceptual decision to complex preferential choice tasks. From early on, they were applied to identification and discrimination tasks (e.g. Ashby, 1983; Edwards, 1965; Heath, 1981; Laming, 1968; Link & Heath, 1975; Pike, 1973); memory retrieval (e.g. Ratcliff, 1978; Stone, 1960; Van Zandt, Colonius, & Proctor, 2000) and classification (e.g., general recognition theory, Ashby, 2000; exemplar-based random walk models of classification, Nosofsky & Palmeri, 1997) to account simultaneously for response times and accuracy data.

They have also been used for preferential decision tasks (e.g. decision field theory, Busemeyer & Townsend, 1993; and multiattribute decision field theory, Diederich, 1997, Diederich & Busemeyer, 1999) and value based decision (Krajbich & Rangel, 2011) to account for choice response times and choice probabilities interpreted as preference strength; judgment and confidence ratings (Pleskac & Busemeyer, 2010); and to account for selling prices, certainty equivalents, and preference reversal phenomena (Busemeyer & Goldstein, 1992; Johnson & Busemeyer, 2005). More recently, they have been applied to combining perceptional

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decision making and preference (e.g. Diederich, 2008; Diederich & Busemeyer, 2006; Gao & Tortell, 2011; Rorie, Gao, McClelland, & Newsome, 2010). Furthermore, these models have been closely linked to measures from neuroscience such as multi-cell electrode recordings, EEG, and fMRI (e.g. Churchland, Kiani, & Shadlen, 2008; Ditterich, 2006; Gold & Shadlen, 2007; Ratcliff, Hasegawa, Hasegawa, Smith, & Segraves, 2007). Under fairly general conditions, these models also represent the optimal rule for making sequentially sampled decisions that balance decision accuracy with cost of sampling (e.g., Bogacz, Brown, Moehlis, Holmes, & Cohen, 2006; Edwards, 1965; Rapoport & Burkheimer, 1971). In practical applications, sequential sampling models have been used to estimate parameters representing basic components of the decision process, such as discriminability, bias, and threshold criterion. Individual differences in these parameters are used to investigate how these parameters differ across age groups, psychopathology, and other populations (e.g. Ratcliff, Thapar, & McKoon, 2010; Thapar, Ratcliff, & McKoon, 2003; White, Ratcliff, Vasey, & McKoon, 2010).

The basic idea of all sequential sampling models is that, when a decision has to be made (a) noisy evidence for or against each choice option is sequentially sampled across time, (b) this evidence is accumulated across time, and (c) a final choice is made as soon as the evidence reaches a threshold, or a deadline has to be met. Choice probability is determined by the probability that evidence level crosses a threshold first for one option before another, and decision time is determined by the time required to reach a threshold. Confidence ratings following a choice can be determined from the strength of evidence that accumulates during a post-choice time interval. There are many specific versions of sequential models that differ according to precisely how evidence is accumulated, how the threshold criteria are set, and how confidence is derived. One class of sequential sampling models assumes that evidence for one option is at the same time evidence against the alternative option. Within this class, random walk models accumulate evidence in discrete time whereas diffusion models accumulate evidence in continuous time. The most commonly used version of the diffusion model is the Wiener diffusion model that linearly accumulates evidence without any decay (Ratcliff, 1978), but others include the Ornstein–Uhlenbeck model that linearly accumulates evidence with decay (Busemeyer & Townsend, 1993; Diederich, 1997), and the leaky competing accumulator (LCA) model (Usher & McClelland, 2001) that nonlinearly accumulates evidence with decay. Another class of sequential sampling models is widespread in psychology: accumulator and counter models. An accumulator/counter is established for each choice alternative separately, and evidence is accumulated in parallel. A decision is made as soon as one counter wins the race to reach one preset criterion. The accumulators/counters may or may not be independent. Poisson-counter models are prominent examples but random walk and diffusion models, one process for each alternative with a single criterion (absorbing boundary) for each process, can also be employed. Other accumulator models such as LATER (linear approach to threshold with ergodic rate) (Carpenter & Williams, 1995) and LBA (Linear Ballistic Accumulator) (Brown & Heathcote, 2005) assume a deterministic linear increase in evidence for one trial. Randomness in responses occurs by assuming a normal distribution across the linear accumulation rate. These models are not considered here further.

In the following we focus on random walk/diffusion models with one process and two decision criteria. For a review of both diffusion models and counter models see Ratcliff and Smith (2004).

Despite the great progress that has been made with the development and empirical testing of random walk/diffusion models, there remain some important limitations. One important limitation of many applications of random walk/diffusion models is that a single integrated source of evidence is assumed to be generating the evidence during the deliberation process leading to a decision. In particular, the integrated source may be based on multiple features or attributes, but all of these features or attributes are assumed to be combined and integrated into a single source of evidence, and this single source is used throughout the decision process until a final decision is reached. There are exceptions developed for very specific applications (e.g. Smith & Ratcliff, 2009; Smith & Sewell, 2013) but by far, single source models predominate the field.

Another limitation is that most random walk/diffusion models cannot account for anticipatory and time-out responses. Trials with a shorter or longer than predefined response time threshold are typically eliminated from the data set.

Finally, most models assume constant decision criteria across the decision process. In some cases, however, it is possible that with elapsed time the boundaries are collapsing, which in neuroscience has been called "urgent signals" (e.g. Churchland et al., 2008; Ditterich, 2006) but see Hawkins, Forstmann, Wagenmakers, Ratcliff, and Brown (2015). We refer also to Zhang, Lee, Vandekerckhove, Gunter, and Wagenmakers (2014) for the inclusion of time-varying boundaries into a single-stage diffusion model.

In the following we will address these topics. To introduce notation, we begin by describing a stochastic process with its relation to psychological concepts. Second, the multi-stage decision model (aka multiattribute attention switching (MAAS) model) is introduced including time and order schedules, finite and infinite time horizons, and non-constant boundaries. Obviously, non-constant boundaries can also be applied to single-stage models. Third, to allow for efficient predictions we discretize the diffusion process (Wiener or Ornstein–Uhlenbeck) by a Markov chain model. Finally, we show the predictions of the model for various scenarios.

## 2. Sequential sampling approach

Sequential sampling models are stochastic processes, that is, a collection of random variables, representing the evolution of some system of random values over time. Two quantities are of foremost interest to psychologists: (1) the probability that the process eventually reaches one of the thresholds or boundaries for the first time (the criterion to initiate a response), i.e., *first passage or exit probability*; (2) the time it takes for the process to reach one of the boundaries for the first time, i.e., *first passage or exit time*. The former quantity is related to the observed relative frequencies, the latter usually to the observed mean choice response times or the observed choice response time distribution.

Let $X(t)$ denote the random variable representing the numerical value of the accumulated evidence at time $t$ (for now we assume that we are in a continuous-time, continuous-state situation). For a binary choice between choice options $A$ and $B$, the models assume that the decision process begins with an initial state of evidence $X(0)$. This initial state may either favor option A ($X(0) > 0$) or option B ($X(0) < 0$) or may be neutral with respect to A or B ($X(0) = 0$), or can be given as a probability distribution.

Upon presentation of the choice options, the decision maker sequentially samples information from the stimulus display over time, retrieves information from memory, or forms preferences, depending on the context. The small increments of evidence sampled at any moment in time are such that they either favor option A ($dX(t) > 0$) or option B ($dX(t) < 0$). The evidence is incremented according to a diffusion process:

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t).$$

Here, $\mu(x, t)$ is called the *effective drift rate* and describes the instantaneous rate of expected increment change at time $t$ and