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Theoretical note

# The first-passage time distribution for the diffusion model with variable drift

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## HIGHLIGHTS

- The Ratcliff diffusion model contains normally distributed drift rates.
- Analytic solution for the cumulative distribution function of the diffusion model.
- The analytical solution is highly efficient and ensures high accuracy.
- Implementations in R statistical language and MATLAB included as online Appendix C.

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ABSTRACT

The Ratcliff diffusion model is now arguably the most widely applied model for response time data. Its major advantage is its description of both response times and the probabilities for correct as well as incorrect responses. The model assumes a Wiener process with drift between two constant absorbing barriers. The first-passage times at the upper and lower boundary describe the responses in simple two-choice decision tasks, for example, in experiments with perceptual discrimination or memory search. In applications of the model, a usual assumption is a varying drift of the Wiener process across trials. This extra flexibility allows accounting for slow errors that often occur in response time experiments. So far, the predicted response time distributions were obtained by numerical evaluation as analytical solutions were not available. Here, we present an analytical expression for the cumulative first-passage time distribution in the diffusion model with normally distributed trial-to-trial variability in the drift. The solution is obtained with predefined precision, and its evaluation turns out to be extremely fast.



## 1. Background

The diffusion model for response times was proposed about 40 years ago (Ratcliff, 1978) as a continuous-time, continuous-state generalization of earlier discrete-time random walk models (Laming, 1968; Link & Heath, 1975). One of its major advantages over standard response time (RT) analyses (i.e., comparison of mean RTs) is the simultaneous analysis of both response time and accuracy. This avoids problems of speed–accuracy trade-offs that are possible confounders of the results and generally difficult to interpret (e.g., Pachella, 1974).

The standard diffusion model assumes a Wiener process with drift $v$ and diffusion coefficient $\sigma^2$ (typically fixed either at $\sigma^2 = 1$ or $\sigma^2 = 0.01$ because it only scales the other parameters) evolving over time in the presence of two absorbing barriers (located at 0 and $a > 0$). Each barrier is associated with one response alternative. The barriers can be viewed as response criteria, that is, the distribution of the first passage time to either barrier produces the predicted response times distribution for the response alternative associated with the barrier.

Although the model is well motivated and the approach is appealing, two issues remain that are often seen as major obstacles for a wider application of the model. Firstly, there is no closed-form solution available for the partial differential equation (PDE) of a diffusion process with the necessary boundary conditions. The available solutions (e.g., Feller, 1968) all require the evaluation of infinite series. These series can be shown to converge quite quickly (Blurton, Kesselmeier, & Gondan, 2012; Gondan, Blurton, & Kesselmeier, 2014; Navarro & Fuss, 2009). However, when fitting the model to data, the series has to be evaluated over and over again, which may take a considerable amount of time. This is

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especially true if more general versions of the model are fitted to data (see the next section). In that case, several numerical integrations have to be carried out that are associated with their own (possibly unknown) approximation errors. However, for parameter estimation, it is useful to have an exact result to avoid numerical problems during estimation (e.g., rough likelihood surfaces).

Secondly, the available solutions only cover the standard Wiener process with constant drift across trials. By analogy to the signal detection model (Tanner & Swets, 1954) and based on common sense arguments (the "resonance" metaphor), Ratcliff (1978) argued that the drift rate $v$ shows inter-trial variability that can be described by a normal distribution: $v \sim N\left(\nu, \eta^2\right)$. For example, one direct consequence of this assumption is that in a response signal paradigm, perceptual sensitivity $d'$ asymptotes and does not reach infinity with signal time $t$ (Ratcliff, 1978, Eq. 10). However, this extra variability comes at the cost of a missing analytical form for the model predictions. Hence, model predictions must be obtained by numerical evaluation instead (Ratcliff & Tuerlinckx, 2002). Interestingly, the *density* function[1] is known for the case of normally distributed drift rates (e.g., Horrocks & Thompson, 2004) and it has been used in the past for fitting the diffusion model to response time data (Ratcliff & Tuerlinckx, 2002; Wiecki, Sofer, & Frank, 2013). For the lower barrier, it is

$$g\left(t \mid \nu, \eta^2, a, w\right) = \frac{1}{\sqrt{t^3\left(1+\eta^2 t\right)}} \times \exp\left[\frac{-\nu^2 t - 2\nu a w + \eta^2 (aw)^2}{2\left(1+\eta^2 t\right)}\right] \times \sum_{j=0}^{\infty} (-1)^j r_j\, \phi\left(\frac{r_j}{\sqrt{t}}\right) \tag{1}$$

where $r_j = ja + aw$ for even $j$ or $r_j = ja + a(1-w)$ for odd $j$, and $\phi(x)$ denotes the standard normal density function evaluated at $x$, and $0 < w < 1$ is the relative starting point of the Wiener process between the two barriers. Without loss of generality the diffusion coefficient $\sigma^2$ has been omitted in (1), as $g'\left(t \mid \nu, \eta^2, \sigma^2, a, w\right) = g\left(t \mid \nu/\sigma, \eta^2/\sigma^2, a/\sigma, w\right)$. The density function is useful if maximum likelihood estimation is desired. However, if parameter estimates are to be obtained from binned data, for example by chi-square methods (e.g., Ratcliff & Smith, 2004) or by the quantile maximum likelihood method (Heathcote, Brown, & Mewhort, 2002), one must rely on numerical integration of the first-passage time density to obtain the distribution function.

Since its introduction additional parameters for inter-trial variability have been added to the model (Ratcliff & Rouder, 1998; Ratcliff & Tuerlinckx, 2002). Thus, the "full" Ratcliff diffusion model fit now requires the numerical evaluation of three integrals (see Tuerlinckx, 2004, Eq. 3). This can become time consuming as the computational complexity raises exponentially (Tuerlinckx, 2004) and all these integrals must be evaluated on infinite series.

Here, we present an analytical solution for the first-passage time distribution of the Ratcliff (1978) model with drift variation. The solution is of theoretical interest and especially for applications of the model. For the application, it increases speed and establishes a pre-defined accuracy of the fitting procedure. It is readily available for use in existing software packages like DMAT (Vandekerckhove & Tuerlinckx, 2008). Researchers that have implemented or seek to implement their own fitting routines will also benefit from the solution as it guarantees a computationally efficient computation with accuracy up to some pre-defined level.

## 2. The cumulative distribution function for the Ratcliff diffusion model

Recently, Gondan and colleagues (2014) reported a solution of the PDE for a Wiener process with constant drift between two absorbing barriers that is using a representation stated in terms of the Mills ratio (Hall, 1997). We would like to remind the reader of some of the favorable properties of this representation. Firstly, it is numerically very stable and no numerical problems arise during the calculation of the infinite series. Secondly, and contrasting its related representation (e.g., Blurton et al., 2012), it is defined for all real drift rates and does not suffer from a singularity at zero drift. Clearly, this is very important when integrating over drift rates. Thirdly, it gives the distribution function and not the survivor function so that the separate calculation of the overall absorption probability at a specific barrier is not necessary. In the most widely adapted representation of the first-passage time cumulative distribution, the survivor function is used. In that case, the series must be subtracted from the probability of terminating at the associated barrier to obtain the cumulative distribution (see Ratcliff, 1978, Eq. A12 and p. 105f, for the motivation of this approach). Obtaining the cumulative directly avoids problems in the derivation regarding this probability with drift variation over trials (see Tuerlinckx, 2004). Apart from the latter issue, these points also hold for the alternative solution that is available and usually used in fitting the diffusion model (Ratcliff, 1978; Ratcliff & Tuerlinckx, 2002). However, the analytic solution for this CDF with inter-trial variability in drift rates is yet unknown.

Using the aforementioned representation (1), the cumulative distribution function $F(t)$ of the first-passage time of a Wiener process with drift $v$ between two absorbing barriers placed at 0 and $a > 0$ and starting at $aw$ $(0 < w < 1)$ to the lower boundary can be expressed by the infinite series (Hall, 1997)

$$F(t \mid v, a, w) = \exp\left(-vaw - \frac{v^2 t}{2}\right) \sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right) \times \left[M\left(\frac{r_j - vt}{\sqrt{t}}\right) + M\left(\frac{r_j + vt}{\sqrt{t}}\right)\right] \tag{2}$$

with $r_j$ and $\phi(x)$ as defined above, and $M(x) = \frac{1-\Phi(x)}{\phi(x)}$ denoting the inverse hazard function (the "Mills ratio") for the standard normal distribution.

In order to obtain a solution for the more general process with trial-to-trial variability in drift rate $v$, one must seek a solution of the integral $\int \psi(x) \cdot F(t \mid x, a, w)\, dx$, that is, one must integrate over the density $\psi(x)$ of the assumed drift distribution and the first-passage time distribution $F(t)$. Because drift rates can take any real value and due to the correspondence with the signal detection model (Tanner & Swets, 1954), the normal distribution is usually chosen as a possible distribution for the drift rates (Ratcliff, 1978, Eqs. 8, A24, & A25). Thus, we replace $\psi(x)$ by the normal density $\phi\left(x \mid \nu, \eta^2\right)$ with mean $\nu$ and variance $\eta^2$. Let $G\left(t \mid \nu, \eta^2, a, w\right)$ be the first-passage time distribution of such a process,

$$G\left(t \mid \nu, \eta^2, a, w\right) := \int_{-\infty}^{\infty} \phi\left(x \mid \nu, \eta^2\right) \cdot F(t \mid x, a, w)\, dx$$
$$= \int_{-\infty}^{\infty} \phi\left(x \mid \nu, \eta^2\right) \exp\left(-xaw - \frac{x^2 t}{2}\right) \sum_{j=0}^{\infty} (-1)^j \phi\left(\frac{r_j}{\sqrt{t}}\right)$$

[1] Note that the distribution (density) is technically not a probability distribution (density) but a defective distribution (density) because it does not integrate to unity. One obtains a proper distribution (density) by summing the distributions (densities) from the upper and lower criteria or by normalizing through the respective absorption probability.