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An APOS study on pre-service teachers' understanding of injections and surjections

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ABSTRACT

This study was undertaken to explore pre-service teachers' understanding of injections and surjections. There were 54 pre-service teachers specialising in the teaching of Mathematics in Grades 10–12 curriculum who participated in the project. The concepts were covered as part of a real analysis course at a South African university. Questionnaires based on an initial genetic decomposition of the concepts of surjective and injective functions were administered to the 54 participants. Their written responses, which were used to identify the mental constructions of these concepts, were analysed using an APOS (action-process-object-schema) framework and five interviews were carried out. The findings indicated that most participants constructed only Action conceptions of bijection and none demonstrated the construction of an Object conception of this concept. Difficulties in understanding can be related to students' lack of construction of the concepts of functions and sets that are a prerequisite to working with bijections.

1. Introduction

Most mathematics education researchers concur that the mathematics education curriculum for pre-service high school teachers in their preparation for teaching should include the opportunity to consolidate mathematically those concepts encountered in the high school mathematics curriculum (Adler, 2002; Ball, Thames, & Phelps, 2008; Wu, 1999). This presents an opportunity to revisit standard topics in the curriculum from an advanced standpoint. The two South African authors have been involved in the planning and teaching of a Real Analysis course for teachers at their university, some aspects of which has been reported in other articles (Bansilal, 2015; Brijlall & Maharaj, 2009). In one article, it was argued that school level concepts in sequences and series could be engaged by pre-service teachers at a deeper level by integrating related ideas in the teaching and learning of monotonic and bounded infinite real sequences (Brijlall & Maharaj, 2009). In this article we continue this approach by examining students' conceptions of functions which are injective and/or surjective (a bijective function is both injective and surjective).

The concept of bijective functions is not part of the school curriculum, it can be seen as an example of what Ball et al. (2008, p.403) call *horizon content knowledge*. This is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum, and is one of the six domains that comprise their model of *mathematical knowledge for teaching* (Ball et al., 2008). Having knowledge of the horizon for mathematics studied at school level, can help teachers make decisions about how to teach concepts like functions. Moreover, the notion of 'inverse' is important in high school and concepts of injectivity and surjectivity

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are central to making sense of the notion of inverse.

The purpose of this article is to examine students' understanding of the concept of injective and surjective functions using Actions, Process, Objects, Schema (APOS) theory as a lens. The research questions are: 1) What APOS mental structures do students develop during the learning of the definitions of injective and surjective functions? and, 2) How can we account for some of the difficulties that students experience in conceptualising injective and/or surjective functions in terms of APOS Theory?.

2. Literature review

Tall and Bakar (1992, p.39) assert that "the concept of a function permeates every branch of mathematics and occupies a central position in its development". There are countless applications of functions to areas such as algebra, trigonometry and mathematical modelling and in everyday situations of mathematics as well. For this reason, there have been many studies which have looked at various teaching and learning issues associated with the concept of functions. Some studies have focused on students' conceptions of functions (Chimhande, 2014; Dubinsky & Wilson 2013; Maharaj, 2010; Thompson, 1994; Tall & Bakar, 1992; Tall, McGowen, & DeMarois, 2000). There are some studies which have examined instructional approaches to teaching functions (Dubinsky & Wilson, 2013; Ronda, 2009; Selden and Selden, 1992); all of them have found that function is an abstract and difficult concept for students. Functions are often presented in the first introduction in algebra as different representations of an Object and studies have also considered the learning and teaching of different representations of functions (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Knuth, 2000; Romberg, Fennema, & Carpenter, 1993; Marcelo and Borba Confrey, 1996; Thompson, 1994). Another aspect studied is related to transformation of functions (Trigueros & Martínez-Planell, 2010) as well as vertical and horizontal shifts of functions (Zaskis, Liljedahl, & Gadowsky, 2003).

While there has been such a wide focus on functions, the concept of surjective and injective functions is underrepresented in existing literature; in fact, we could not find any research studies focusing on this concept. Hence this study aims to fill this gap by looking at pre-service teachers' understanding of injective and surjective functions. Tirosh & Tsamir (1996) did carry out a study on one-to-one correspondence justifications. They presented fourteen problems on infinite sets to 189 10th to 12th grade middle school learners. They found that the participant justifications were mainly displayed by numerical, explicit and geometrical representations.

Since the construction of the concepts of interest is closely related with students' ability to construct formal arguments we also reviewed some literature about proofs. Moore (1994) unpacked the strategy used by the instructor in his study to explain why a function was onto. The instructor began with an informal example, then drew upon the definition, then gave a second informal example before finally explaining how to use the definition in a proof. Moore commented that having a repertoire of examples at hand helps a person to move seamlessly between concept images, definitions and usage of definitions in constructing proofs. Students on the other hand struggled because they had a "limited repertoire of domain-specific knowledge from which to pull examples" (Moore, 1994, p.260). Moore also found that students were often stuck at the beginning of a proof, sometimes starting with an incorrect hypothesis but once they got some help on how to proceed they were able to complete the proofs. Moore articulates that the reason for many of the students' difficulties with the definitions, was that the concepts were seen as too abstract. Students had difficulty creating mental pictures of the concepts and without this informal understanding they were not able to learn the written form of the definition. However even though examples and informal approaches were helpful to them, this did not mean that it was sufficient for them to write a correct proof. Moore (1994) asserts that the process of generating one's own examples demands cognitive skills different from those involved in studying examples given by the instructor or the textbook.

Weber (2001, p.111) emphasizes the role of "strategic knowledge" in deriving proofs. This refers to heuristic guidelines that are used to recall actions most likely to be useful, or to select a particular action from possible alternatives that could be applied. Weber (2001, p.105) uses the term "syntactic knowledge" to describe the knowledge of facts required to construct the proof. However he found that students often had the syntactic knowledge but were unable to apply this to derive the proof. Weber (2001) found that more than half of the undergraduate students' failed attempts were because they failed to apply the syntactic knowledge. When the students were specifically told to use certain facts required in the propositions, they were able to construct the required proof. Weber (2001) asserted that the primary cause of undergraduates' failure to construct a proof was likely to be limited strategic knowledge.

3. Theoretical framework

Piaget (1985) expanded and deliberated on the notion of reflective abstraction as involving two steps, the first of which involves reflection on existing knowledge. The second step involves a projection of existing knowledge into a higher plane of thought. Piaget (1985) and Dubinsky et al. (2005a, 2005b) attribute the construction of logico-mathematical structures by a learner during the Process of cognitive development to reflective abstraction. Dubinsky's belief that reflective abstraction could serve as a powerful tool to describe the mental development of more advanced mathematical concepts, led to the development of APOS theory.

The mental structures of Action, Process, Object and Schema constitute the acronym APOS. APOS theory postulates that a mathematical concept develops as one tries to transform existing physical or mental Objects. Although the progression from Action to Process and to Object may be considered as a development of knowledge levels, learning does not follow a specific direction. It includes partial developments between structures and moving back and forth between them. It may thus be possible that different students may deal with the same mathematical situation in a different way depending on the structures they have constructed. The descriptions of Action, Process, Object and Schema given below are based on those given in Arnon et al. (2014).

Action: A transformation is first conceived as an *Action*, when it is a reaction to stimuli which an individual perceives as external. It consists in following specific external or internal instructions, and the need to perform each step of the transformation explicitly.