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"Approximate" multiplicative relationships between quantitative unknowns

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ABSTRACT

Three 18-session design experiments were conducted, each with 6–9 7th and 8th grade students, to investigate relationships between students' rational number knowledge and algebraic reasoning. Students were to represent in drawings and equations two multiplicatively related unknown heights (e.g., one was 5 times another). Twelve of the 22 participating students operated with the second multiplicative concept, which meant they viewed known quantities as units of units, or two-levels-of-units structures, but not as three-levels-of-units structures. These students were challenged to represent multiplicative relationships between unknowns: They changed the given relationship, did not think of the relationship as multiplicative until after concerted work, and used numerical values in lieu of unknowns. Our account for these challenges is that students needed to simplify the involved units coordinations. Ultimately students abstracted the relationship as multiplicative, but the exact relationship was not certain or had to be constituted in activity. Implications for teaching are explored.

1. Introduction

As increasing numbers of students take algebra courses in middle and high school (Nord et al., 2011; Rampey, Dion, & Donahue, 2009; Stein, Kaufman, Sherman, & Hillen, 2011), algebra teachers are tasked with working with a greater diversity of students. Teaching a greater diversity of students in algebra has been managed in several ways, from putting lower-skilled students into double periods of algebra (Nomi & Allensworth, 2013) to teaching all students in heterogeneous groups with supports such as a 2-year algebra course for all students (Boaler & Staples, 2008) or extra workshops and after school help (Burris, Wiley, Welner, & Murphy, 2008). In whatever ways student diversity is managed, more needs to be known about the algebraic thinking and learning of a wide range of secondary students in order to inform the kind of supports that both students and teachers need.

One way in which students differ in learning algebra is in the meanings (Thompson, 2013) they develop for key algebraic ideas such as unknowns, variables, equivalence, equations, and functional relationships. The field of mathematics education has largely turned its attention to functional relationships as a basis for algebra, where unknowns are thought of as an outcome of "freezing" variation (e.g., Carraher & Schliemann, 2007; Chazan, 2000). However, much is still not known about how to help students build productive meanings for unknowns and use algebraic notation to represent relationships between unknowns (Kieran, 2007), whether the situations originate in freezing variation or not. As Kieran has stated, "Generating equations to represent the relationships found in typical word problems is well known to be an area of difficulty for algebra students" (p. 721).

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In a project about how to differentiate instruction for cognitively diverse middle school students, we conducted three 18-episode design experiments. One purpose was to study how to differentiate instruction (Hackenberg, Creager, Eker, & Lee, 2016); another purpose was to investigate relationships between students' rational number knowledge and algebraic reasoning. The purpose of this paper is to describe and account for the challenges that 12 of the 22 students across the experiments faced in representing multiplicative relationships between two unknown heights with drawings and equations.[1] These students were operating with what we call the second multiplicative concept (MC2 students, Hackenberg and Tillema (2009), discussed in Section 3.2). MC2 students can take two levels of units as given, which means that they can think of numbers and lengths as units of units, or composite units. For example, they can conceive of a height partitioned into, say, five equal parts without having to actually make the partitions. In contrast, students who operate with the third multiplicative concept can think of numbers and lengths as units of composite units (three levels of units), and students who are operating with the first multiplicative concept cannot take as given a length as a composite unit. In our experiments the remaining 10 of the 22 students were MC3 students. Including students operating at two different levels of multiplicative reasoning in the experiments is a main reason we refer to the participating students as cognitively diverse. In addition, the students demonstrated differences in fractional knowledge as assessed at the start of each experiment via a one-on-one interview and written worksheet (discussed in Section 4; see Appendix A and Appendix B).

An MC2 student in experiment 1, Tim,[2] surprised us in how he worked on these problems. In representing problems where an unknown corn stalk height was five times an unknown tomato plant height, Tim insisted that the relationship was *approximately* five times. Although not all MC2 students used this term, their activity in representing multiplicative relationships between quantitative unknowns indicated to us that the relationships were challenging for them to structure. Our research questions are:

1) How do MC2 students represent multiplicative relationships between two quantitative unknowns?
2) Why do MC2 students have difficulty structuring multiplicative relationships between two quantitative unknowns?

To respond to these questions, we present relevant parts of our second-order models of the MC2 students. A second-order model is a researcher's constellation of constructs to describe and account for another person's ways of operating (Steffe, von Glasersfeld, Richards, & Cobb, 1983). We consider these models to be one tool that can help teachers make informed decisions about how to tailor instruction to students who are cognitively diverse (Hackenberg et al., 2016; Ulrich, Tillema, Hackenberg, & Norton, 2014). So, the models are a key tool in our research to understand how to differentiate instruction.

2. Algebraic reasoning, notating algebraically, and students' ways of thinking

To frame our study we review literature on approaches to algebraic reasoning, notating algebraically, and students' ways of thinking about unknowns, variables, and equations.

2.1. Approaches to algebraic reasoning

Although some researchers have suggested a fundamental disconnect between students' arithmetical and algebraic ways of thinking (e.g., Balacheff, 2001; Filloy & Rojano, 1989; Heffernan & Koedinger, 1998; Lee & Wheeler, 1989), many researchers emphasize algebraic reasoning as arising from students' generalizations of and abstractions from their arithmetical and quantitative reasoning (e.g., Carraher, Schliemann, Brizuela, & Earnest, 2006; Empson, Levi, & Carpenter, 2011; Kaput, 2008; Smith & Thompson, 2008; Stacey, 1989). For example, focusing on quantities in working algebraically with students can be useful because quantities can be thought about and represented without specific values (Smith & Thompson, 2008). Following these researchers, we view students' beginning algebraic reasoning to be about (1) generalizing and abstracting arithmetical and quantitative relationships, and systematically representing those generalizations and abstractions, not necessarily with standard algebraic notation; and (2) learning to reason with algebraic notation in lieu of quantities. These two points correspond to Kaput's (2008) two core aspects of algebra, with a focus on the first of his three strands.[3]

We take *generalizing* to be "an activity in which people in specific sociomathematical contexts engage in at least one of three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases" (Ellis, 2011, p. 311). We take *abstracting* to be a process that underlies generalizing, that involves a de-emphasizing of particulars and an ability to view them as representing a wider phenomenon (Kaput, Blanton, & Moreno, 2008; Piaget, 2001; von Glasersfeld, 1991, 1995).

2.2. Notating algebraically

Many researchers describe active symbolization as nested and cyclical: Students' representations (verbal, gestural, drawn, written) of their experiences mediate a refined interpretation of those experiences, which in turn produces a refined set of

[1] We focused on height because middle school students have had substantial experience with discrete quantities in elementary school (discussed further in Sections 2 and 3), and because height is one of the most basic continuous quantities (Piaget, Inhelder, & Szeminska, 1960).

[2] All student names are pseudonyms; all students discussed are seventh-grade students unless noted.

[3] Kaput's (2008) second strand is functions, relations, and variation, and his third strand is modeling.