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Covariational reasoning among U.S. and South Korean secondary mathematics teachers☆,☆☆

Patrick W. Thompson*, Neil J. Hatfield, Hyunkyoung Yoon, Surani Joshua, Cameron Byerley

Arizona State University, Colorado State University, United States

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ABSTRACT

We investigated covariational reasoning among 487 secondary mathematics teachers in the United States and South Korea. We presented an animation showing values of two varying magnitudes (v and u) on axes in a Cartesian plane along with a request that they sketch a graph of the value of u in relation to the value of v. We classified teachers' sketches on two independent criteria: (1) where they placed their initial point, and (2) their graph's overall shape irrespective of initial point. There are distinct differences on both criteria between U.S. and South Korean teachers, suggesting that covariational reasoning is more prominent among South Korean secondary teachers than among U.S. secondary teachers. The results also suggest strongly that forming a multiplicative object that unites quantities' values is necessary to express covariation graphically.

Copur-Gencturk (2015), Zaslavsky (1994), and Thompson (2013) argued compellingly that how teachers understand a mathematical idea is an important factor in the mathematical understandings that students actually form. The more coherently teachers understand an idea they teach, the greater are students' opportunities to learn that idea coherently. Inversely, the less coherently teachers understand an idea they teach, the fewer are students' opportunities to learn that idea coherently.

A number of studies support the claim that reasoning covariationally is a powerful foundation for students' comprehension of many mathematical ideas. Students' ability to reason covariationally supports their understanding of:

- proportion (Karplus, Pulos, & Stage, 1979; Karplus, Pulos, & Stage, 1983; Lobato & Siebert, 2002),
- rate of change and linearity (Adu-Gyamfi & Bossé, 2014; Castillo-Garsow, 2013; Confrey, 1994; Herbert & Pierce, 2012; Nunes, Desli, & Bell, 2003; Thompson and Thompson, 1996; Thompson, 1994a,c; Zaslavsky, Sela, & Leron, 2002),
- variable (Clement, 1989; Dogbey, 2015; Goldenberg, Lewis, & O'Keefe, 1992; Hitt & González-Martín, 2015; Montiel, Vidakovic, & Kabael, 2008; Schoenfeld & Arcavi, 1988; Thompson & Carlson, 2017; Trigueros & Jacobs, 2008; Trigueros & Ursini, 1999; Trigueros & Ursini, 2003; Yerushalmy, 1997),
- trigonometry (Moore, 2012, 2014; Thompson, Carlson, & Silverman, 2007)
- exponential growth (Castillo-Garsow, 2013; Confrey, 1991, 1994; Confrey & Smith, 1994, 1995; Ellis, Özgür, Kulow, Williams, & Amidon, 2012; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Ellis, Özgur, Kulow, Dogan, & Amidon, 2016)
- functions of one and two variables (Boyer, 1946; Bridger, 1996; Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002;

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* Correspondence to: P. O. Box 871804, Tempe, AZ 85287-1804, United States.
E-mail address: pat@pat-thompson.net (P.W. Thompson).

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Confrey, 1992; Hamley, 1934; Hitt & González-Martín, 2015; Kaput, 1994; Keene, 2007; Martínez-Planell & Gaisman, 2013; Nemirovsky, 1996; Thompson, 1994a, 1994b; Thompson & Carlson, 2017; Weber & Thompson, 2014; Yerushalmy, 1997).

These same bodies of literature, involving small numbers of subjects, suggest that reasoning covariationally is uncommon among students and teachers, at least in the U.S. Also, subjects in these studies were drawn from geographic locales. We know of no studies that investigate covariational reasoning either internationally or with a large, geographically diverse sample.

1. Theoretical background

Our meaning of covariational reasoning is grounded in the mental operations described by Thompson's theory of quantitative reasoning (Thompson, 1993, 1994c, 2011). In this theory, a quantity exists only to the extent that someone conceives it, so the nature of any quantity is idiosyncratic to the individual conceiving it. That a person has conceived a quantity means that she has conceived an attribute of some object in a way that it is measurable. The person need not know an actual measure of the attribute, but she takes for granted that it has one and understands what it means.[1] Accordingly, a quantity's value varies when the person conceiving it envisions that the object's attribute varies and hence that the attribute's measure varies.

We characterize quantities as being idiosyncratic to the person conceiving them, for many reasons. One reason is that this removes the onus that we must describe quantities only in terms of the most sophisticated conceptions held by experts. We are free to characterize *learners*' quantities—their conceptions of objects' attributes and their quantification—as differing in principle from experts' conceptions. Most importantly, we are free to describe the quantities and relationships that individuals have conceived as opposed to describing what they have misconceived, or what an expert might say they do not conceive.

Saldanha and Thompson (1998) described mature covariational reasoning in terms of quantities whose values vary:

> Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image (Saldanha & Thompson, 1998, p. 299).

Saldanha and Thompson's idea of multiplicative object derives from Piaget's notion of logical multiplication—an operation that Piaget and Inhelder described as underlying multiple classification and seriation, and more generally as underlying relationships of simultaneity (Inhelder & Piaget, 1964, p. 182). A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one *and* the other. As noted by Thompson and Saldanha (2003), conceptualizing torque as a physical quantity is one example of forming a quantity by uniting attributes of an object (lever plus fulcrum in this case). The attribute "amount of twist" is conceived as being constituted simultaneously by a rotational force and a distance from the fulcrum at which the force is applied.

The ability to form multiplicative objects is at the heart of understanding mathematical ideas of function, rate of change, accumulation, vector space, and so on. Also, although the science education literature does not leverage the idea of forming a multiplicative object of two quantities' attributes, we see students' ability to form multiplicative objects as being central to their understandings of many physical quantities, e.g. force, work, momentum, energy, and so on.

Saldanha and Thompson's emphasis on the centrality of multiplicative objects in a person's ability to reason covariationally gains support from Stalvey and Vidakovic's (2015) investigation of 15 Calculus 2 students' attempts to envision height and volume of water simultaneously in each of two containers as they emptied. Stalvey and Vidakovic reported that a majority of students struggled to envision values of height and values of volume as varying simultaneously in order to sketch a graph of one in relation to the other. Students could envision general directions of the covariation (e.g., height decreases as volume decreases), which Thompson and Carlson (2017) termed *gross coordination of values,* but they could not reason about both height and volume varying simultaneously over small intervals of change. Stalvey and Vidakovic reported that students could attend to height sans volume or to volume sans height, but they struggled to attend to volume and height simultaneously. Carlson et al. (2002) reported similar results in their study of calculus students' abilities to reason covariationally about dynamic situations described textually.

Saldanha and Thompson (1998) did not mention graphs when they spoke of "the entailing correspondence being an emergent property of the image" of covariation. Rather, in line with Goldenberg (1988, 1993) and Goldenberg et al. (1992), they spoke about a person's covariation scheme as entailing an overall image, in retrospect or anticipation, of the simultaneous states of two quantities as they vary. Graphs are a common way to represent this image.

Likewise, Kaput (1994) and Thompson and Carlson (2017) characterized mathematicians' early, pre-graphical conceptions of function relationships as entailing an image of covarying quantities. The genius of Descartes' method of graphs is that, for a person who reasons covariationally, a graph produced within the conventions of a coordinate system provides a visualization that captures what Saldanha and Thompson termed "the entailing correspondence" between values of two covarying quantities. Unfortunately, in

[1] We speak of quantities' measures or values throughout this paper even though speaking of quantities' magnitudes often would be more appropriate. The distinction between magnitude and value or measure is important, but discussing it in here would not serve the paper's purpose. See (Thompson, Carlson, Byerley, & Hatfield, 2014) for a full discussion of measures and magnitudes.