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Proof validation and modification in secondary school geometry

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ABSTRACT

Proof validation is important in school mathematics because it can provide a basis upon which to critique mathematical arguments. While there has been some previous research on proof validation, the need for studies with school students is pressing. For this paper, we focus on proof validation and modification during secondary school geometry. For that purpose, we employ Lakatos' notion of local counterexample that rejects a specific step in a proof. By using Toulmin's framework to analyze data from a task-based questionnaire completed by 32 ninth-grade students in a class in Japan, we identify what attempts the students made in producing local counterexamples to their proofs and modifying their proofs to deal with local counterexamples. We found that student difficulties related to producing diagrams that satisfied the condition of the set proof problem and to generating acceptable warrants for claims. The classroom use of tasks that entail student discovery of local counterexamples may help to improve students' learning of proof and proving.

1. Introduction

Proof and proving are recognized as playing a key role in shaping meaningful mathematical experiences for all students (e.g., Stylianides, Stylianides, & Weber, in press). One major area of proof research in mathematics education is research on the reading of proofs. As suggested by Weber (2015), empirical studies on the reading of proofs can be broadly classified into three categories. A first category encompasses studies in which different types of mathematical argument (visual, inductive, generic, or deductive) are presented to students and the students are asked to evaluate whether the arguments are personally convincing and whether the arguments can be considered to constitute proofs (e.g., Healy & Hoyles, 2000; Martin & Harel, 1989; Segal, 1999). A second category relates to students' understanding of given correct proofs. Based on models of proof comprehension (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Yang & Lin, 2008), studies in this category seek to identify effective proof comprehension strategies and examine the effectiveness of specific training to improve students' proof comprehension (Hodds, Alcock, & Inglis, 2014; Samkoff & Weber, 2015). A third category involves proof validation; here, researchers show students purported deductive proofs and ask them to determine whether the proofs are valid or invalid (Alcock & Weber, 2005; Inglis & Alcock, 2012; Ko & Knuth, 2013; Selden & Selden, 2003; Weber, 2010).

This paper reports on a study that belongs to the third category, proof validation. We use the term proof validation to mean the reading of arguments constructed as proofs to check whether the arguments really constitute legitimate proofs; that is, whether the

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arguments can establish the truth of mathematical statements (see Section 3 for a more elaborate definition).

Although there have been studies of proof validation behavior, the participants in many of those studies consisted of undergraduates, trainee or in-service teachers, and/or mathematicians. The need for studies involving school students is pressing, especially given recent curriculum reforms in several countries indicating that proof validation should be introduced into secondary school mathematics. For example, the latest national curriculum in England declares that, as part of mathematical reasoning, "pupils should be taught to […] assess the validity of an argument and the accuracy of a given way of presenting information" (Department for Education, 2014; pp. 5–6). In the United States, the Common Core State Standards Initiative (2010) also underlines the importance of constructing viable arguments and critiquing the reasoning of others and places these activities among the standards for mathematical practice. As Inglis and Alcock (2012) argue, "clearly, one way of critiquing explanations is by validating purported proofs" (p. 360). Moreover, Powers et al. (2010) show that explicit teaching of proof validation has the potential to improve students' capacity to construct valid proofs. Research into proof validation may offer an approach to addressing the difficulties frequently encountered by students attempting to construct valid proofs (e.g., Healy & Hoyles, 2000; Hoyles & Healy, 2007).

To introduce proof validation into secondary school mathematics, it is vital to explore the nature of the validation attempts made by secondary school students and their subsequent modifications to proofs. This is because if mathematics educators can deepen their understanding of students' behavior in this regard, such understanding can provide a basis for designing more effective teaching and for helping students become more effective at proof validation and modification. Hence, in this paper, we aim to explore this issue (more specific research questions are given in Section 3 below) using data from a task-based questionnaire completed by 32 ninth-grade students in a Japanese secondary school class.

The structure of the remainder of this paper is that in Section 2 we review the literature on proof validation. In Section 3, we define in detail the respective meanings of proof validation and proof modification, and specify the research questions addressed in this paper. We account for our research method in Section 4, and then present and discuss our analysis of the data with Toulmin's framework in Sections 5 and 6. We conclude in Section 7 by summarizing our study and indicating suggestions for future research.

2. Research on proof validation

To the best of our knowledge, Selden and Selden (1995, 2003) first introduced the term *proof validation* into mathematics education research (although Segal, 1999; and Knuth, 2002b; investigated relevant behavior in undergraduates and teachers without using the term). Selden and Selden (2003) conducted a study in which undergraduates were given four arguments, including both valid and invalid ones, and were asked to judge whether each of them could be regarded as a valid proof. The researchers found that although only about half of judgments made by the participants were correct at the initial stage, the rate of correct judgments improved after the interviewer's encouragement to reflect on their initial judgments. Another finding was that many students incorrectly accepted as a valid proof an argument showing the truth of the converse of a target proposition, indicating that they tended to judge the validity of arguments based on local details rather than global/structural points.

Since Selden and Selden's (2003) seminal work, several researchers have investigated how undergraduates and mathematicians work on proof validation. Alcock and Weber (2005) looked at line-by-line checking of an argument, observing that only a few undergraduates detected a hidden warrant in the step from a certain line of the argument to the subsequent line and recognized the invalidity of the warrant. Weber (2010) focused on failure reasons in proof validation by undergraduates, hypothesizing possible causes including students' tendency to overlook the inappropriateness of assumptions used in an argument, and limitations in students' own mathematical knowledge. Ko and Knuth (2013) considered arguments from various content domains (in algebra, analysis, geometry, and number theory) and observed that undergraduates' judgment on the validity of the arguments varied according to content domain. Ko and Knuth also dealt with ostensible counterexamples proposed to show the falsity of propositions. In contrast to these studies involving undergraduates, Weber (2008) and Inglis and Alcock (2012) explored how experts (mathematicians) engaged in proof validation and how their behavior was different from that of undergraduates.

Although these existing studies certainly provide multiple insights into proof validation behavior, there are two issues that remain open. First, that a majority of existing studies regarding proof validation involve undergraduates, trainee and in-service teachers, and/or mathematicians, means that the need for studies of proof validation in school mathematics is pressing. In doing so, we focus on the domain of geometry because it is a key area in which proofs and proving are learned and used in secondary school mathematics. While there have been numerous studies on proof and proving in secondary school geometry covering students' capabilities regarding proofs and their perspectives on proving in classes (Herbst & Brach, 2006; Hoyles & Healy, 2007; Senk, 1985), teachers' conceptions of proof (Knuth, 2002a, 2002b), textbook analysis (Fujita & Jones, 2014; Otten, Gilbertson, Males, & Clark, 2014), task design (Cirillo & Herbst, 2012; Komatsu, in press), and classroom-based research, including investigation of student-teacher interactions and teaching interventions to enhance student learning (Martin, McCrone, Bower, & Dindyal, 2005; Miyazaki, Fujita, & Jones, 2015), only a few studies have related to proof validation, mostly focusing on whether students can discern the invalidity of circular arguments (in which conclusions are used as suppositions). For instance, McCrone and Martin (2004) surveyed 18 American high school students and showed that only 22% of the students correctly judged a circular argument invalid. A similar result was obtained by Reiss et al. (2001), who administered the proof questionnaire by Healy and Hoyles (1998) to German secondary school students. More recently, Miyazaki, Fujita, and Jones (2017), using their own framework to capture students' understanding of the structure of proofs, analyzed a classroom episode where the invalidity of a circular argument was discussed among students; the researchers make reference to a component of their framework, namely hypothetical syllogism, to explain student inability to reject the circular argument. However, because proof validation does not only involve circular arguments, it remains necessary to address other types of proof validation.