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Drawing inferences from learners' examples and questions to inform task design and develop learners' spatial knowledge

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ABSTRACT

Examples that learners generate, and questions they ask while generating examples, are both sources for inferring about learners' thinking. We investigated how inferences derived from each of these sources relate, and how these inferences can inform task design aimed at advancing students' knowledge of *scale factor enlargement* (i.e. scaling). The study involved students in two secondary schools in England who were individually tasked to generate examples of scale factor enlargements in relation to specifically designed prompts. Students were encouraged to raise questions while generating their examples. We drew inferences about students' thinking from their examples and, where available, from their questions. These inferences informed our design and implementation of a set of follow-up tasks for all students, and an additional personalised task for each student who raised any questions. Students showed increased knowledge of, and confidence with, scale factor enlargement independently of whether they asked questions during the exemplification task.

1. Introduction

Getting insight into students' mathematical thinking processes during their engagement with tasks can help teachers decide where next to develop students' knowledge. There is, however, a fundamental difficulty: individuals' thinking processes are not directly observable. Although this is a barrier for any teacher seeking to understand the way their students think, research in mathematics education indicates that inferential analysis of learners' self-generated examples can help alleviate this difficulty (Sandefur, Mason, Stylianides, & Watson, 2013; Sinclair, Watson, Zazkis, & Mason, 2011; Watson & Mason, 2005; Watson & Shipman, 2008; Zazkis & Leikin, 2007).

In this article we consider not only *learner generated examples* (LGEs) as a source for inferring about learners' thinking, but also the questions learners raise while generating their examples. We argue that by cross-examining inferences drawn from a student's examples with those drawn from their associated questions, initial inferences made from example productions alone can be enhanced. Hence, when inferring students' knowledge, the combined consideration of example- and question-based inferences may better inform future lesson strategies than inferences drawn from examples alone.

We related the example- and question-based inferences we drew to two aspects of students' thinking, which we view as synergistic: (1) *conceptual knowledge,* which we interpret as knowledge of relationships between mathematical features (Hallett, Nunes, & Bryant, 2010; Hiebert & Lefevre, 1986), and (2) *procedural knowledge,* which we interpret as methodological knowledge underlying how mathematical outcomes are arrived at (Byrnes, 1992; McCormick, 1997). A generic but useful way of capturing the

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distinction between these two knowledge types is to view conceptual knowledge as knowledge *that*…, and to view procedural knowledge as algorithmic knowledge of *how to*… (Byrnes, 1992; Davis, 1983).

Seeing mathematical knowledge as a combination of both these knowledge types is consistent with the view that conceptual and procedural knowledge can be inherently integrated (Gray & Tall, 1994; Skemp, 1987; Star, 2007). Indeed, as we explain later, in our study we found it appropriate to attribute a combination of the two knowledge types to the majority of learners. Yet we found that attributing such combinations necessitated prior delineation of underlying conceptual and procedural knowledge based on the definitions we provided earlier.

By considering conceptual and procedural knowledge separately we were able to identify not only where and how both knowledge types co-existed, but also use their separate constructs to help us consider and articulate more precisely the nuances of the mathematical knowledge we deemed a learner to have. For instance, we found useful the constructs of procedural knowledge in situations when a learner's example or question implied use of a specific "heuristic" (Star, 2005) or an order of steps; we found useful the constructs of conceptual knowledge when a learner's example implied how the meaning of, or connectivity to, a particular feature was derived. Star and Stylianides (2013) suggest that giving attention to these aspects of conceptual and procedural knowledge increases the chances that other researchers can use, test, and build on resultant findings. Further, the language of these within-type knowledge distinctions offers operational applicability across different mathematical domains (e.g. algebra, geometry, arithmetic, etc.), something that Star and Stylianides (2013) state is important if unified use of the terms 'conceptual knowledge' and 'procedural knowledge' is to be achieved across levels of education.

We used our inferences of students' knowledge to inform the design of follow-up tasks aimed at advancing students' understanding of scale factor enlargement. In this study we use the term *scale factor enlargement* (common in the English curriculum) as a synonym for *scaling*. To facilitate our aim of advancing students' knowledge, we also considered how the examples a student generated allowed for characterising and extending their enlargement-related personal example space. Following Watson and Mason (2005), we define a *personal example space* (PES) as the inferred set of mathematical objects and construction techniques each individual can bring to mind in response to an exemplification task.

We chose scale factor enlargement to research exemplification as it was an up-and-coming curriculum topic for the two groups of secondary (11–12 year olds) in England which participated in the study. Scale factor enlargement was also a topic that both groups had yet to encounter in secondary school. Introducing new mathematical concepts to learners through use of LGEs is described by Watson and Shipman (2008) as having developmental plausibility for all learners and the potential to lead to significant learning. Further, the pedagogy of scale factor enlargement has hitherto attracted little research attention (Bryant, 2009) and thus our study contributes to filling the gap in research on children's spatial learning. Therefore, this research responds to a range of unmet needs and was guided by the following research questions:

(1) What inferences can be drawn about students' conceptual and procedural knowledge of scale factor enlargement based only on the *examples* they generate?
(2) What inferences can be drawn about students' conceptual and procedural knowledge of scale factor enlargement based only on the *questions* they ask?
(3) How do example- and question-based inferences of students' conceptual and procedural knowledge compare?
(4) What impact does solving tasks, which are designed on the basis of all available inferences, have on students' knowledge of, and confidence with, scale factor enlargement?

Before discussing the methods employed to address these questions, we review key prior research in the area of exemplification. We also consider how the act of questioning can be seen as integral to example generation and thus to inferences of learners' mathematical thinking.

2. Research on exemplification and questioning

The role of examples as a tool to introduce a new concept can be traced back as far as Aristotle's trilogy *Rhetoric*. A huge arc can be drawn straight from his narrative that 'example resembles induction, and induction is a beginning' (Freese, 1926, trans.) to recent research on using LGEs to introduce new mathematical concepts (Watson & Mason 2005; Watson & Shipman, 2008). According to some researchers, everything can be seen as an example of something so long as a relationship is perceived (Goldenberg & Mason, 2008; Sinclair et al., 2011). In regard to the role of examples in mathematics education, Watson and Mason (2005) distinguish between the broad notion of example *use*, e.g. in textbooks and by teachers, and *exemplification* which involves the employment of examples to represent mathematical generalities within a given context. This description of exemplification fits with our use of the term in this article because students were expected to generalise, i.e. to notice consistency, in mathematical objects present within a range of prompts we provided to introduce them to the topic of scale factor enlargement.

The importance of LGEs in mathematics education appears in two interlinked domains: the role example use can play in increasing learners' knowledge (Charles, 1980; Sandefur et al., 2013; Stylianides and Stylianides, 2009; Watson & Mason, 2005), and the insights researchers and teachers can gain into learners' thinking. Regarding the latter domain, the insights include where learners focus their attention and the way they reason with what they see as relevant (Sinclair et al., 2011; Stylianides & Stylianides, 2009; Watson & Mason, 2005; Zazkis & Leikin, 2007).

Why and how learners make mathematical connections, enabling them to classify and generalise using examples, are central to structuring inferences about their possible understandings (Zazkis & Leikin, 2007; Sinclair et al., 2011; Goldenberg and Mason, 2008).