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Integer comparisons across the grades: Students' justifications and ways of reasoning☆

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ABSTRACT

This study is an investigation of students' reasoning about integer comparisons—a topic that is often counterintuitive for students because negative numbers of smaller absolute value are considered greater (e.g., $-5 > -6$). We posed integer-comparison tasks to 40 students each in Grades 2, 4, and 7, as well as to 11th graders on a successful mathematics track. We coded for correctness and for students' justifications, which we categorized in terms of 3 ways of reasoning: magnitude-based, order-based, and developmental/other. The 7th graders used order-based reasoning more often than did the younger students, and it more often led to correct answers; however, the college-track 11th graders, who responded correctly to almost every problem, used a more balanced distribution of order- and magnitude-based reasoning. We present a framework for students' ways of reasoning about integer comparisons, report performance trends, rank integer-comparison tasks by relative difficulty, and discuss implications for integer instruction.

1. Introduction

Integers are an important and challenging topic in the transition from arithmetic to algebra (Peled & Carraher, 2007). Children first learn about whole numbers, which are rather intuitive because they relate to counting and quantifying sets of items in the world. Against this backdrop, the notion of a negative number—often described as being "less than zero"—requires some suspension of disbelief. In many everyday contexts, such as numbers of people, toys, cookies, and so on or measures such as length or area, the idea of a number less than nothing seems absurd. How then do students make sense of integers? In particular, what kinds of justifications do they offer for their judgments that one integer is greater than or less than another?

We interviewed 40 students each in Grades 2, 4, 7, and 11 and asked them to compare pairs of integers, such as -7 and 3. In this paper, we report on the justifications that students offered for such comparisons; we categorize the justifications as belonging to broader ways of reasoning about integers. We compare and contrast students' reasoning by looking both across problems and across grade levels. In this way, we answer how students reason about different cases of integer comparisons

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and identify trends in the reasoning of students with different levels of familiarity with negative numbers. We conclude with implications for integer instruction that are informed by trends in students' reasoning.

In reviewing the literature, we found a lack of literature that systematically sampled K-12 students at different grade levels and documented their explicit reasoning about integer comparisons. We present a framework of students' ways of reasoning and associated justifications for integer comparisons. We also report compelling trends in students' reasoning both within and between groups. Given the counterintuitive nature of negative numbers, understanding students' thinking about them is particularly important for supporting student learning. The findings reported here advance the field's understanding of students' thinking about integers. Such research contributes to the efforts of the mathematics education research community to support instruction that enables students to successfully transition from arithmetic to algebra (e.g., Moses & Cobb, 2002; Peled & Carraher, 2007).

2. Theoretical perspective

We approach this study from a children's mathematical-thinking perspective (Steffe, 1991). We regard children's mathematical thinking as being different from that of adults in interesting and important ways. We take seriously the nature of children's mathematics, whether or not it is correct from an expert perspective. We believe that seeing mathematics through children's eyes is important for better understanding the sense that they make. This perspective is based on constructivist principles that children have existing knowledge and experiences they bring with them into the classroom and upon which they continue to build (e.g., Carpenter, Fennema, Franke, Levi, & Empson, 1999; Fuson, Smith, & Lo Cicero, 1997; Steffe & Olive, 2010; Steffe, 1991, 2002, 2004). We take this view because the ultimate goal of our research is to find ways to better support children's learning of mathematics (Carpenter et al., 1999; Carpenter, Franke, & Levi, 2003; Empson & Levi, 2011).

3. Background

In the early elementary grades, students become acquainted with whole numbers and the basic operations involving these. While they progress in their mathematical educations, students encounter different kinds of numbers. Whole-number arithmetic is relatively intuitive for children because they can reason about it in ways that are grounded in real-world contexts (e.g., Carpenter et al., 1999). (Nonnegative) fractions, decimals, and percentages present additional challenges, because children are asked to reason about numbers with values between the whole numbers, including between 0 and 1. Furthermore, these numbers are represented in many ways, and children must learn to relate the various representations. At the same time, nonnegative rational numbers can still be related to amounts in real-world contexts (e.g., 3/4 of an apple pie). As they do with whole numbers, children can draw on their intuitions and physical experiences to make sense of fractions and to represent them in multiple ways (e.g., Empson & Levi, 2011). With the introduction of negative numbers, new challenges arise.

In the United States, instruction on integer arithmetic is typically concentrated in middle school (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010; Whitacre et al., 2011). By that time, students' comfort levels with nonnegative numbers may work against them (Bruno & Martinón, 1999). They are asked to expand their mathematical worlds to include negatives, as well as to reconceive of familiar numbers as positive (Whitacre et al., 2016). Previous generalizations cease to be true, and the bounds of mathematical reality are challenged. For example, children who think of −7 as representing 7 of something are asked to see −7 as *less than* 3, but how can 7 of something be less than 3 of something? Understandably, the introduction of integers involves notions that are counterintuitive for children (e.g., Vlassis, 2004).

3.1. Magnitude and order

When considering the pedagogical challenges of supporting her students' understanding of negative numbers, Ball (1993) wrote,

> Any number has two components: magnitude and direction; from a pedagogical point of view, this seems to become particularly significant when the students' domain is stretched to include negative numbers. A focus on the magnitude component leads to a focus on absolute value. This component emerges prominently in many everyday uses of negative numbers (e.g., debt, temperature). Thus, comparing magnitudes becomes complicated. There is a sense in which −5 is more than −1 and equal to 5, even though, conventionally, the "right" answer is that −5 is less than both −1 and 5. This interpretation arises from perceiving −5 and 5 as both five units away from zero and −5 as more units away from zero than −1. Simultaneously understanding that −5 is, in one sense, more than −1 and, in another sense, less than −1 is at the heart of understanding negative numbers. (p. 379).

Historically, mathematicians struggled to resolve the contradiction between magnitude and order in the interest of consistency (Gallardo, 2002; Henley, 1999). This dilemma was resolved by the adoption of a more abstract notion of number and a convention that privileges order over magnitude. In mathematics today, the symbols < and > and the corresponding terms *less than* and *greater than* refer to comparisons of order rather than magnitude. In other words, the statements "2 > −10"