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An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts

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ABSTRACT

Much research on calculus students' understanding of applied derivatives has been done in kinematics-based contexts (i.e. position, velocity, acceleration). However, given the wide range of applications in science and engineering that are not based on kinematics, nor even explicitly on time, it is important to know how students understand applied derivatives in non-kinematics contexts. In this study, interviews with six students and surveys with 38 students were used to explore students' "ways of understanding" and "ways of thinking" regarding applied, non-kinematics derivatives. In particular, six categories of ways of understanding emerged from the data as having been shared by a substantial portion of the students in this study: (1) covariation, (2) invoking time, (3) other symbols as constants, (4) other symbols as implicit functions, (5) implicit differentiation, and (6) output values as amounts instead of rates of change.

1. Introduction

As calculus education research has grown in the last few decades, there has been much attention given to student understanding of the derivative concept in pure mathematics contexts (e.g., Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Baker, Cooley, & Trigueros, 2000; Habre & Abboud, 2006; Park, 2015; Siyepu, 2013). However, the derivative concept is not only important within mathematics, but is also a useful concept in other disciplines like physics, engineering, economics, chemistry, and biology. Unfortunately, the few studies from the science and engineering education perspectives that have discussed students' understanding and application of the derivative in their own disciplines have described that students struggle in using this concept (Beichner, 1994; Bucy, Thompson, & Mountcastle, 2007; Christensen & Thompson, 2012; McDermott, Rosenquist, & van Zee, 1987; Sazhin, 1998). In mathematics, the derivative can have many meanings, such as the slope, the limit of the difference quotient expression, or a rate of change (Zandieh, 2000). In calculus curriculum, however, the graphical "slope" meaning seems to be given the largest amount of attention (Briggs, Cochran, & Gillett, 2015; Stewart, 2014; Thomas, Weir, & Hass, 2009). Even in texts that use applications to introduce the derivative (e.g., Hughes-Hallett et al., 2012), the bulk of the development of the derivative concept is done through the idea of slope. While science and engineering may also adopt or make use of the slope meaning, derivatives in these contexts tend to be focused most on the "rate of change" meaning (Bingolbali, Monaghan, & Roper, 2007). In fact, special words are often given to rates of change in science and engineering, such as velocity for change in distance per change in time, power for change in work per change in time, linear density for change in mass per change in length, or elasticity for change in demand per change in price. Note

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that in these examples, "rate of change" does not have to be time-dependent, but can refer to how much any one quantity might change as another quantity changes (see Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998). The mismatch between slope-based interpretations in mathematics classrooms and rate-of-change interpretations in science and engineering classrooms may contribute to some difficulties students have (Bingolbali et al., 2007).

Because of the derivative's importance in science and engineering fields of study and the difficulties encountered by students in using it in those fields, some calculus education researchers have begun to give more attention to how students understand, apply, and use the derivative concept in contexts outside of mathematics (e.g., Berry & Nyman, 2003; Gerson & Walter, 2008; Roorda, Vos, & Goedhart, 2007, 2010; Roschelle, Kaput, & Stroup, 2000). However, the vast majority of the mathematics education research dealing with applications of rate of change and the derivative is centered on the contexts of position, velocity, and acceleration (e.g., Berry & Nyman, 2003; Bezuidenhout, 1998; Bowers & Doerr, 2001; Hale, 2000; Marrongelle, 2004; Petersen, Enoch, & Noll, 2014; Roschelle et al., 2000; Schwalbach & Dosemagen, 2000; Thompson, 1994a; Zandieh, 2000). In this paper, the contexts of position, velocity, and acceleration are together referred to as "kinematics," which makes use of the traditional physics definition of "kinematics" as the movement of physical objects (see Serway & Jewett, 2008). While velocity and acceleration are certainly common and useful applications of the derivative, there are myriad other uses of this concept in science and engineering (e.g., Gerson & Walter, 2008), including many applications that are not time-dependent (e.g., Roorda et al., 2007). Furthermore, velocity is such an intuitive concept that it might not require students to fully unpack the meaning of a "rate of change," which researchers have found that calculus students might not even completely understand (Bezuidenhout, 1998; Byerley, Hatfield, & Thompson, 2012; Thompson, 1994a, 1994b). In other words, it is possible that kinematics applications may not fully develop the rate of change meaning of the derivative that is so critical to many contexts in science and engineering.

The general focus on kinematics in applications of the derivative highlights the need for calculus education research to explore other possible applications of the derivative that are not kinematics based, and perhaps not even time based. While a few studies have considered individual cases of non-kinematics applications of the derivative, they have often been focusing on particular theoretical aspects of mathematical thinking, namely blending conceptual spaces (Gerson & Walter, 2008) and mathematical modelling (Roorda et al., 2007). There is still much unknown about general student understanding and thinking about a range of applied, non-kinematics derivatives. Exploratory studies at this level are needed to investigate more generally the potential cognitive demands, issues, or difficulties that may arise for students in working with an array of these kinds of derivatives. Research along these lines could help illuminate the barriers that might exist for students in effectively applying the derivative concept to science and engineering. In response to this need, this study aims to identify what are termed students' *ways of understanding* and *ways of thinking* (Harel & Sowder, 2005; Harel, 2008a, 2008b), defined in detail in Section 3, in connection to these types of derivatives. Using this terminology, this exploratory study on student understanding of derivatives can be summarized with the following research question: What are common ways of understanding and ways of thinking students exhibit when working with applied, non-kinematics derivatives?

2. Research on kinematics-focused derivatives

While this study is focused on an exploration of student understanding of non-kinematics derivatives, it is helpful at this point to discuss the research literature on kinematics derivatives understanding for reference and comparison. I use this literature base to examine relationships that may exist to the results found in this study. The literature on kinematics derivatives can be seen as containing three major topics: (1) student difficulties and misconceptions, (2) student blending of kinematics and derivatives, and (3) teaching strategies for developing student understanding. I briefly describe each in the following paragraphs.

In the first topic, several authors have discussed issues students face as they try to make sense of kinematics derivatives. For example, Bezuidenhout (1998) noted that students tended to overgeneralize the idea that applied derivatives are either velocity or acceleration. He had given students the function $S(v)$, which denoted the stopping distance, S, of a car that began braking at the velocity, v. Many students interpreted the derivative, $S'(v)$, as "acceleration," stating that for a given v, S' would denote the acceleration of the car. Further, it appears that student difficulties in conceptualizing rates of change goes deeper than simply mistaking a derivative of a function involving velocity as always representing acceleration. Thompson (1994a, 1994b) noted that students often did not fully unpack "rates," like velocity, into their constituent parts of distance divided by time. He described students' tendencies to "interpret [velocity] as 'how fast it [the function] is changing,' without interpreting the details of the expression as an amount of change in one quantity in relation to a change in another" (p. 246). In other studies, both Berry and Nyman (2003) and Hale (2000) mentioned the difficulty some students might have at times confusing the function value with the derivative value. Hale stated that when asked to give the slope (i.e. velocity) of a position graph at a given point, students may instead respond with the graph's height, which she suggested may be due to insufficient conceptualizations of the kinematics variables themselves (i.e. position, velocity, and acceleration).

In the second topic, because of the close relationships between the calculus concept of the derivative and the physics concept of kinematics, some researchers have attempted to identify how students might draw on kinematics in order to make sense of derivatives. For example, Marrongelle (2004) created a classification for how students might invoke kinematics in discussing derivatives. She found that some students ("contextualizers") completely merged the two contexts and thought of them as, essentially, equivalent. Other students ("example-users") occasionally invoked the physics of kinematics to frame the problem, but would reason mostly within purely mathematical ideas. Other students avoided using kinematics