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# Dylan's units coordinating across contexts

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ABSTRACT

Units coordination has emerged as an important construct for understanding students' mathematical thinking, particularly their concepts of multiplication and fractions. To explore students' units coordination development, we conducted an eleven-session constructivist teaching experiment with a pair of sixth-grade students, investigating how they coordinated whole number and fractional units in discrete and continuous settings. In this paper we focus on one student, Dylan, who reasoned with whole number units but not fractional units at the beginning of the teaching experiment. We describe Dylan's development of units coordination as he continued to reason with whole number units in fractional situations, and we discuss implications for instruction.



## 1. Introduction

Many elementary curricular standards now include an early focus on conceptions of fractions as measures (Lamon, 2007) alongside conceptual understanding of whole number arithmetic. For instance, the United States' *Common Core State Standards for School Mathematics* includes both "represent and solve problems involving multiplication and division" and "understanding fractions as numbers [that can be plotted on a number line]" as third-grade objectives (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 21). Researchers have identified a shared cognitive necessity for understanding fractions and for conceptualizing multiplication (and division) – an ability to *coordinate multiple levels of units* (Hackenberg & Tillema, 2009; Steffe, 1992; Steffe & Olive, 2010).

Units coordination has emerged as an important construct for understanding mathematical thinking after elementary school as well. These domains include students' writing of linear equations involving unknown quantities (Hackenberg & Lee, 2015; Olive & Çağalan, 2008), students' ways of operating additively with signed quantities (such as integers; Ulrich, 2012), students' combinatorial reasoning (Tillema, 2014), and teachers' interpretations of fractions representations (Izsák, Jacobsen, de Arajuo, & Orhill, 2012). Meaningful attainment of middle and secondary school learning goals is likely to continue to present a challenge to teachers and students, as research suggests that many students enter sixth-grade yet to coordinate multiple levels of units (Boyce & Norton, 2016; Hackenberg, 2013; Hackenberg & Lee, 2015; Norton & Boyce, 2013).

In this paper, we report on an investigation of how engagement in different mathematical situations might foster middle-grades students' development of units coordination. More specifically, we consider an integrated development of whole number concepts and fractions concepts, with students' development of structures that apply to both situations as an underlying objective. We focus our analysis on a single sixth-grade student, Dylan, who participated in paired-student constructivist teaching experiment (Steffe & Thompson, 2000). We analyze Dylan's units coordination as he engaged with

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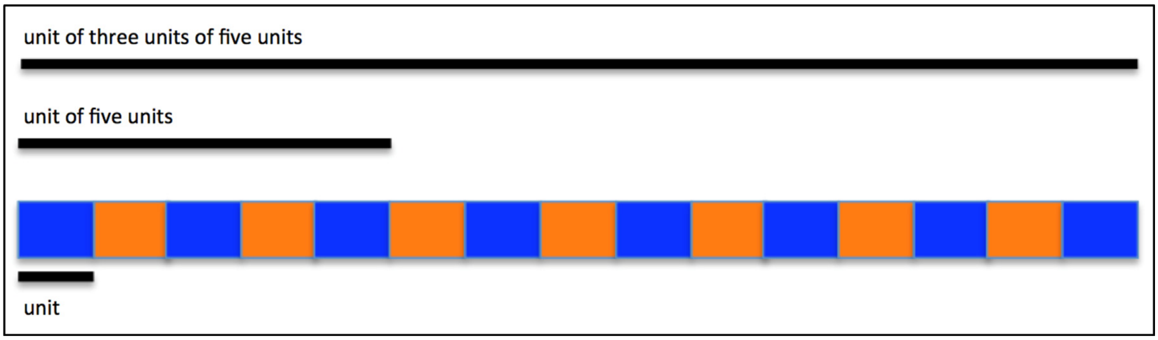


**Fig. 1.** Representation of a structure for coordinating units.

fractions and whole number tasks in discrete and continuous settings. We describe how Dylan progressed in his units coordination development even as he consistently approached fractions tasks by reasoning with whole number units instead of fractional units.

## 2. Theoretical and conceptual framing

We adopted a scheme theoretic perspective for our study (von Glasersfeld, 1995). In scheme theory, mental activity in service of a goal begins with the recognition of a salient situation; i.e., an individuals' fitting of a situational goal to a *scheme* for which a sequence of available mental actions (*operations*) are expected to yield a satisfactory result (von Glasersfeld, 1995). Steffe (2001) describes units coordination as the "mental operation of distributing a composite unit across the elements of another composite unit" (p. 279). He describes how this operation is involved in a child's scheme for counting.

> The result of the units coordination permits the child to experience a unit containing five units of two that is not simply a sensory-motor experience. This organized experience is what I regard as a situation of the scheme. The activity of the scheme in the case of the example is to count by two five times to specify the numerosity of the individual units contained in five units of two, and the results of the scheme is the experience of the immediate past counting activity along with its result. (Steffe, 2001; p. 279).

Modifications to a scheme can affect one or more of its three constituent parts: the recognition template, the operations, or the expected result. For instance, students who are *perturbed* by a novel situational goal or surprising outcome may form an accommodation to successfully fit the situation within an existing scheme or abduct novel activity in an attempt to form a new way of operating in that situation (Norton, 2008 von Glasersfeld, 1995). It is via the process of *reflective abstraction* that a student's scheme transitions from relying on mental or physical activities with sensory material to becoming *interiorized* and anticipatory (Piaget, 2001; Simon & Tzur, 2004; Simon, Placa, & Avitzur, 2016).

Consider the following Nickels task: You have three nickels (5-cent pieces); how many more nickels would you need to have 35 cents? Upon hearing the words "three nickels," a student might immediately know (*assimilate*) that three 5 s is also 15 1 s. This is an example of *interiorized* coordination of two *levels of units*. The coordination may be *reversible* in the sense that a student might anticipate computing other quantities in terms of *either* 5 s or 1 s before learning the rest of the task. Another student might arrive at the correct response to the Nickels task, but require actions such as tallying or counting—units coordinating *in activity*—to coordinate the three 5 s as fifteen 1 s.

We incorporate aspects of Piaget's structuralism (1970) to characterize *organizations* of interiorized, reversible operations. Fig. 1 represents the form of a structure for coordinating three levels of units — for knowing 15 as a result of composing fifteen 1 s, three 5 s, or one 15. A units coordinating structure is a way of operating within such a form (Norton, Boyce, Phillips et al., 2015).

### 2.1. Stages of units coordinating activity

As students' develop more powerful schemes, they progress through stages characterized by the types units coordinating activity in which they engage (Norton, Boyce, Ulrich, & Phillips, 2015; Steffe & Cobb, 1988). Along with decreased reliance on physical activity or perceptual material, students develop flexibility in their ability to form composite units and an ability to reflect on their reasoning with inverse relationships as they progress from Stage 1 (in which children require activity to form a numerical composite) to Stage 2 (in which children can assimilate a unit composed of other units and further compose or decompose units in activity) to Stage 3 (in which children can assimilate with units within units within units) (Hackenberg, 2010; Norton, Boyce, Ulrich et al., 2015).