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The learning and teaching of linear algebra: Observations and generalizations

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ABSTRACT

This paper is about a teaching experiment (TE) with inservice secondary teachers (hereafter "participants") in the theory of systems of linear equations. The TE was oriented within particular social and intellectual climates, and its design and implementation took into consideration a series of findings concerning the difficulties students have in linear algebra. The questions we set for this study were: (1) Did the participants in the particular TE climates construct viable knowledge in the theory of systems of linear equations? Our criteria for viable knowledge consist in evidence for the ability to (a) generate non-trivial conjectures, judged so subjectively by a mathematician, (b) prove such conjecture, and (c) move upward along the APOS conception levels. (2) What difficulties and insights did the participants experience as they constructed such knowledge?

The potential contributions of our investigation into these questions to researchers and practitioners include (a) a detailed depiction of the participants' achievements and challenges in dealing with theoretical questions concerning linear systems in an authentic learning environment and under a tutelage oriented in a particular constructivist perspective; and (b) a field-based hypothesis about the consequences of a particular learning environment vis-à-vis construction of knowledge in linear algebra.

All of the participants had taken a linear algebra course as part of their undergraduate studies, on average 17 years prior to the TE, with an average grade of about 80%. Thus, a third question set for this study concerns retention. (3) What did the participants retain from their linear algebra courses vis-à-vis concepts, ideas, and problem solving pertaining to the theory of systems of linear equations, assuming they had constructed such knowledge during these courses?

This paper is about a teaching experiment (TE) with inservice secondary teachers (hereafter "participants"), in the theory of systems of linear equations. All of the participants had taken a linear algebra course as part of their undergraduate studies. The TE was not conducted with a particular goal to investigate learners' construction process of or difficulties with a specific concept or a cluster of concepts in linear algebra, as typically is the case in research on the learning and teaching of this area. Rather, the TE was conducted holistically with two goals in mind.

The first goal was two-fold: (a) to document the participants' retention of linear algebra material pertaining to the theory of systems of linear equations, and (b) to document the development of the participants' conceptualizations of this theory in a particular learning environment characterized, generally, by its focus on collaborative learning and uncompressing attention to the mathematical integrity of the content taught and learners' intellectual need.

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The second goal was to generalize the observations made in the TE into *field-based hypotheses* on the learning and teaching of elementary linear algebra in general. A field-based hypothesis is a tentative answer to a research question suggested by observations of learners' mathematical behaviors in an authentic learning environment, and is explained by cognitive and instructional analyses oriented within a particular theory of learning, but has not, yet, been proved or disapproved by rigorous empirical methodologies in large scale settings.[1] In this respect, the results of the TE are not *findings* in the strict sense of the term, but field-based observations germane to the two goals of the study.

Collectively, the two goals call for the following tasks: (a) describe the learning environment in which the TE was oriented, (b) articulate the theory of learning underpinning the instructional intervention employed in the TE, and (c) state the research questions targeted by the TE. Research-based findings on the sources of student difficulties in linear algebra were among the critical inputs that regulated the planning and implementation of the TE; they are outlined in Section 1. Items (a) and (b) are the subject of Section 2, whereas Item (c) is the focus of Section 3. Section 4 presents the methodology used for the TE's data collection and analysis. Sections 5 and 6, the bulk of the paper, outline the two instructional units (Unit 1 and Unit 2) planned for the TE and the observations made during their implementation. Each of these two sections is organized around three parallel Sections 5.1–5.3 and 6.1–6.3 as follows:

- Sections 5.1 and 6.1 discuss the planning stages of Unit 1 and Unit 2, respectively.
- Sections 5.2 and 6.2 discuss the observations made.
  - o Section 5.2 is organized in six Sections 5.2.1–5.2.6, corresponding to the six observations made in relation to Unit 1.
  - o Section 6.2 is organized in 10 Sections 6.2.1–6.2.10, corresponding to 10 observations made in relation to Unit 2.
- Sections 5.3 and 6.3 are summaries of the observations made in relation to Unit 1 and Unit 2, respectively.

Section 7 concludes with the contributions of the study and the field-based hypotheses generated from these observations.

1. Sources of student difficulties

Linear algebra is one of two courses—the other being calculus—required by virtually all science, engineering, economics, and mathematics students, the number of whom is likely to be in the hundreds of thousands. Yet, by all accounts—subjective reports by mathematicians who taught linear algebra as well as research studies—students find linear algebra difficult. An example of the subjective reports is Carlson's (1993) broadly-cited reflection on his experience of teaching elementary linear algebra to undergraduate students, in a paper entitled "Teaching linear algebra: Must the fog always roll in?" He writes:

> [T]he second half of the title of this paper refers to something that seems to happen whenever I teach linear algebra. My students first learn how to solve systems of linear equations, and how to calculate products of matrices. These are easy for them. But when we get to subspaces, spanning, and linear independence, my students become confused and disoriented. It is as if a heavy fog has rolled in over them, and they cannot see where they are or where they are going. And I, as their teacher, become disheartened, and question my choice of profession. (p. 29).

Mathematicians who taught elementary linear algebra are likely to identify with Carlson's experience. Carlson proposes several reasons for the difficulties students have with linear algebra: lack of mathematical maturity; inadequate experience in dealing with concepts (as opposed to computations); often different algorithms are required to solve the same problem appearing in different contexts (e.g., the procedure applied to finding a basis for a row space is different from that applied to finding a basis for a vector space of functions); central concepts, such as linear combination, linear independence, and subspace, are introduced rapidly over a short period of time and are disconnected from the students' prior experiences. Some of these observations, though subjective, are backed up by research, as we will see shortly.

Many students fail linear algebra courses or leave them with profoundly negative experience (Robert & Robinet, 1989). Even students who were judged to have completed linear algebra courses successfully, by measures of their grades, fail to retain (or perhaps have never properly constructed) even the most basic concepts. Harel (1998) studied a group of 25 undergraduate students one-to-three semesters after they had completed two courses: a semester-long course in differential equations—a substantial portion of which was devoted to linear algebraic topics, such as determinants and eigen theory—followed by a semester-long course in elementary linear algebra. The average grade of these students in each of the two courses was approximately 75%. On the other hand, the distribution of the students' correct answers to a list of simple problems involving basic linear-algebraic concepts was strikingly low; and most of their responses were impoverished or incomprehensible.

The obstacles students encounter in learning linear algebra are a hybrid of what Brousseau (1997) dubs didactical obstacles—obstacles caused by narrow or faulty instruction—and epistemological obstacles—obstacles that are unavoidable due to the abstraction level of the discipline. In this section, we outline such obstacles, focusing on those that are of a general nature rather than specific to a particular linear algebraic concept, and state briefly the actions taken in the design and implementation of the TE to address them. These actions will come into a fuller view as the paper unfolds.

[1] This definition is consistent with the use of the term "hypothesis" in science, in that it is a proposition not yet verified but set forth to explain certain facts or phenomena in light of established theories.