Journal of Mathematical Behavior 44 (2016) 34–49

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Improving calculus explanations through peer review

Daniel Lee Reinholz

Department of Mathematics & Statistics, San Diego State University, 5500 Campanile Drive, San Diego, CA 92182, United States

ARTICLE INFO

Article history:
Received 9 May 2016
Received in revised form
30 September 2016
Accepted 2 October 2016

Keywords:
Explanation
Reflection
Peer assessment
Calculus

ABSTRACT

This paper describes Peer-Assisted Reflection (PAR), a peer-review activity designed to help students explain mathematics. PAR was implemented in a single experimental section during two consecutive semesters (phases) of introductory college calculus. During the second semester (Phase II), students were explicitly taught how to provide better feedback to each other. As a result, the amount and quality of feedback provided by Phase II students was significantly improved from Phase I. During both phases, students who used PAR had significantly better explanations than a comparison group that did not include PAR, indicating that student explanations can be improved with relatively little intervention. To capture these improvements, I introduce a new analytic scheme that defines explanation as a cluster concept along four dimensions. This context-neutral scheme operationalizes explanations in a way that growth can be captured longitudinally.

1. Introduction

Mathematical understanding is multi-faceted. To engage proficiently with mathematics, students must: master concepts and procedures, engage strategically in problem solving, explain and reflect on their work, and develop productive dispositions towards knowing and doing mathematics (NRC, 2001). In particular, the sociocultural turn in mathematics describes learning as a process of being enculturated in appropriate social practices (Engle, 2012; Lave, 1996). Accordingly, the present study focuses on Peer-Assisted Reflection (PAR), an activity designed to support the development of a particular practice, explanation.

Each week, students completed a draft solution to a PAR homework problem and engaged in a structured peer review process. PAR created space for students to reflect on their explanations, exchange feedback with their peers, and incorporate their learning into improved explanations through revision. This paper focuses on the impact of PAR on student explanations, extending prior work that demonstrated PAR's impact on student success (Reinholz, 2015b).

PAR was developed over a multiple phase design experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Collins, Joseph, & Bielaczyc, 2004). Two of these phases took place in calculus, a conceptually rich (Oehrtman, Carlson, & Thompson, 2008; Tall, 1992) and challenging (Bressoud, Carlson, Mesa, & Rasmussen, 2013) area of mathematical inquiry. These phases are the focus of the present work. To capture changes in student explanations over the course of a semester, I introduce a new analytic scheme. This scheme provides a lens for quantifying changes in the quality of student explanations, designed to work across mathematical content domains. Additionally, this paper discusses the impact on the types of feedback students provided to each other as a result of being taught how to provide better feedback during Phase II. This paper aims to make two

E-mail address: daniel.reinholz@sdsu.edu

http://dx.doi.org/10.1016/j.jmathb.2016.10.001

primary contributions to the literature: (1) it documents the impact of a new instructional technique on improving student explanations, and (2) it provides a content-neutral analytic scheme for capturing changes in the quality of explanations.

2. Theoretical framing

2.1. Defining explanation

Explanation plays a central role in a variety of disciplines, including: science (Braaten & Windschitl, 2011), mathematics (CCSSM, 2010), and psychology (Lombrozo, 2006). Explanations, as defined by philosophers of science, generally focus on describing *how* or *why* something happens (Braaten & Windschitl, 2011). In mathematics, explanation is considered an important aspect of proof (De Villiers, 2003; Harel & Sowder, 2007; Steiner, 1978), and valued in its own right. Psychological studies (Lombrozo, 2006) highlight the cognitive function of explanations in generalizing understandings, which helps promote learning (Chi, De Leeuw, Chiu, & LaVancher, 1994; Wong, Lawson, & Keeves, 2002).

Given its centrality as a mathematical practice and as a tool for learning, standards documents from across the world elevate explanation. In the US, the *Principles and Standards for School Mathematics* include communication as one of five mathematical process standards (NCTM, 2000). Similarly, the Danish KOM project dedicates the *communication competency* to explanation (Niss, 2011), and explanations are central to the reasoning strand in the Australian Curriculum (ACARA, 2009). The US *Common Core State Standards for Mathematics* even prize explanation as a "hallmark" of understanding (CCSSM, 2010).

Although there is consensus on the value of explanation, defining explanation is a challenge. For instance, science educators continue to debate the relationship between explanation and argumentation (Osborne & Patterson, 2011). Similarly, philosophers of science have yet to agree on a common definition of explanation (Wilson & Keil, 1998). Thus, while there is general consensus that explanation is valuable, there is less agreement on exactly what explanation is. I argue that proof offers a way forward for mathematics educators. Given the centrality of proof to mathematics, defining proof has received considerable attention (Weber, 2014). The progress the field has made in defining proof is instructive when it comes to explanation.

To define proof, educators have attempted to create a set of desirable criteria that all proofs should meet. However, there is no single shared understanding of what constitutes a proof in the mathematical community. While some proofs would unambiguously be classified as proofs by nearly all professional mathematicians (e.g., rigorous deductive arguments), there are many *proofs* where classification is up for debate (e.g., graphical or computation proofs). As such, it has been nearly impossible to distill an "essence of proof" that can be embodied into a set of criteria for classification purposes.

To overcome this difficulty, proof can be understood as a cluster concept (Lakoff, 1987; Weber, 2014). To define a cluster concept, one begins with a set of criteria related to proofs. When a particular proof meets all of the criteria, it is unambiguously classified as a proof. When a proof satisfies only some of the criteria, it is a borderline case that may be considered a proof by some and not by others. This accounts for the ambiguity with socially-defined concepts, but still allows for the community to move forward by identifying a set of desired characteristics for proofs. Not intended to be exhaustive, a subset of criteria one might consider includes: (1) proof as a "convincing argument, as judged by qualified judges" (Hersh, 1993), (2) proof as a social process, meaning proofs must be sanctioned by the mathematical community (Harel & Sowder, 2007; Mason, Burton, & Stacey, 2010), and (3) proof as an explanation of why something is true (Hanna, 1990).

In analogy, explanations can be defined as a cluster concept. The first feature of explanations is that they must explain something, which is called the 'explanandum' (Lombrozo, 2006). In a proof, a *key idea* (Raman, 2003) contains the essential elements required to construct the proof, but falls short of representing a proof per se. Key ideas for explanations are analogously defined as describing some essential component of the mathematics in a problem, but are incomplete in themselves. In this sense, a key idea provides the raw material from which a complete explanation can be constructed. In explaining mathematics, an individual must express their private understanding of this key idea so that it can be conveyed to others publicly. Because communication is social, this process is context-dependent.

Although explanations must be judged in context, they still must conform to the larger body of knowledge established in the field of mathematics, related to the *accuracy* of an explanation. Accuracy concerns whether or not an explanation is in some sense "right," regardless of how illuminating or easy to follow it is. In addition to being accurate, mathematical explanations must be *precise*, using language that is exact. This can be achieved by correctly using the standard lexicon of mathematics (Jones, 2000), the appropriate use of colloquial language, or even in inventing new definitions. Perhaps the most fundamental aspect of an explanation is that it should describe *how* or *why* something happens (Braaten & Windschitl, 2011). In mathematics, proofs are considered to play two distinct roles; they both *prove* and *explain* (De Villiers, 2003; Steiner, 1978). A proof that *proves* demonstrates *that* a statement is true (related to *accuracy*); a proof that *explains* also makes clear *why* a statement is true (Hanna, 1990). This idea of describing why is denoted as *justification*. Finally, visual representations are a useful way to develop and express mathematical thinking (Cuoco, Goldenberg, & Mark, 1996; NCTM, 2000). Visual representations, or *diagrams*, often embody understandings that may be difficult to express in words.

Taken together, the quality of an explanation can be measured along four dimensions: accuracy, precision, justification, and diagrams. As a cluster concept, would one generally expect that explanations of higher quality would be scored higher along each dimension, although there may be some ambiguous cases in which a high-quality explanation scores low on some