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Moving in and out of contexts in collaborative reasoning about equations

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ABSTRACT

This case study investigates how a group of 12-year-old pupils contextualizes a task formulated as an equation expressed in a word problem. Of special interest is to explore in detail the phenomenon of pupils working with manipulative-based equation-solving methods in a task involving another real world context. The pupils' small group discussions were videotaped and analyzed in terms of how the pupils contextualized the task in their attempts to arrive at an answer. The results highlight the importance of giving pupils opportunities to realize the particular position of symbolic mathematical representations when dealing with mathematical concepts. While an abstract concept describes something general, concrete representations and specific real-world examples always describe something specific. No one particular example incorporates the rich meaning of an abstract concept. This central distinction needs to be included in teaching practices.

1. Introduction

Equations constitute one of the central concepts in school mathematics and in mathematics more generally. In their simplest form, they are deceivingly close to arithmetic, but learning about equations is also the entry point to algebra and other, more advanced, forms of mathematics. When mathematics is used to model problems from everyday life, science or other ostensibly non-mathematical phenomena, equations are almost always involved. In applied mathematics, entire fields are formed around specific equations and how to use these equations, either theoretically or computationally. Regardless of whether we look at school mathematics or Nobel Prize for physics or economics, equations can be found at the intersection between concrete and abstract phenomena and mathematical ideas. The close connection with everyday situations, on the one hand, and symbolic abstract mathematics, on the other hand, makes equations interesting from an educational perspective. Different approaches to teaching algebra, such as solving equations using concrete examples, introducing functional situations, or modelling physical phenomena, have been proposed, and considerable research interest has been devoted to such issues (Bednarz, Kieran, & Lee, 1996). It is interesting to consider the effects on learning that we can expect from different teaching approaches, and equations can also serve as typical cases of much more general issues in the teaching of mathematics—a way of connecting the world of concrete objects and events to abstract mathematical ideas and reasoning.

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The connection between the concrete and the abstract[1] was theorized by prominent 20th-century developmental and educational psychologists, such as Piaget (e.g., 1970), Vygotsky (1934 /2012) and Dewey (1910 /1997). Bruner (1966) presented a hierarchical order from enacted to iconic to symbolic representations. In mathematics education research, knowledge and concepts are often seen as simultaneously concrete and abstract (Radford, 2014). The connection of understanding one in terms of the other is described as a two-way process in both directions (Filloy & Rojano, 1989), along a concrete-abstract continuum (Siler & Willows, 2014), with simultaneous movements from the abstract to the concrete, and *vice versa* (Roth & Hwang, 2006). Lesh (1981) discusses these connections in more detailed terms, as connections between several representations: manipulatives, real-world situations, pictures, written symbols and spoken words. The phenomenon is often dicussed in terms of semiotics, constituting learning in mathematics as characterized by processing or converting representations, where the conversion might involve moving between registers, e.g., from verbal or iconic to algebraic or geometric symbolism (Duval, 1995; Winsløv, 2004). Therefore, various specific instructional strategies have been suggested to support the transfer from concrete to abstract representations. Examples include Heddens' (1986) model of a continuum from the concrete through the semi-concrete and semi-abstract to the abstract level. Another example is an explicit concrete-representational-abstract (CRA) sequence (Witzel, Mercer, & Miller, 2003); a third is "concreteness fading", which is also a three-step progression, where the concrete instantiation becomes increasingly abstract over time (Fyfe, McNeil, Son, & Goldstone, 2014).

Specific concrete objects can be designed (e.g., Cuisenaire rods, geoboards, tangrams, 3-dimensional blocks, and interlocking cubes) or assembled from everyday life (e.g., beans, matchboxes, tooth sticks, strings, and empty bottles) for educational purposes to "represent explicitly and concretely mathematical ideas that are abstract" (Moyer, 2001; p. 176). Such objects are usually called manipulatives because they have both "visual and tactile appeal and can be manipulated by learners through hands-on experiences" (ibid.). In mathematics education, the use of manipulatives has been discussed for decades (Suh & Moyer, 2007; Suydam & Higgins, 1977). Koedinger, Alibali and Nathan (2008) consider the question of whether concrete representations are "generally better or worse than abstract representations" a "false dichotomy" (p. 389), and Cooper (2012) argues that learning does not only depend on whether manipulatives are used but also on *which* manipulatives are supplied and *how* they are utilized. In line with these studies, Fyfe et al. (2014) hopes that research will move beyond this dichotomy. Current research is shifting towards a more balanced approach when examining the use of concrete objects (Uttal, O'Doherty, Newland, Hand, & Loache, 2009), investigating the different contexts and representations that may facilitate learning and how they do so (Koedinger et al., 2008; Marley & Carbonneau, 2014).

The study presented here aims to deepen our understanding of the role of manipulatives in pupils' learning with equations. However, studies about manipulatives and the learning of mathematics, often focus on situations where the pupils are explicitly instructed to work with objects, which they can touch and move around. There is a lack of studies that examine when pupils, on their own initiative, invoke experiences of manipulatives in their work with mathematical tasks. The reason for lack of attention to this central issue is, most likely, because it is difficult to *a priori* find such situations.

The present study presents an in-depth analysis of a videotaped sequence showing a group of 12-year-old pupils working together on a task involving an equation presented in a word problem. The particular sequence is interesting because of the way that the pupils use methods from earlier instruction and incorporate their experiences with manipulatives as a resource when working on the task, even though they have not been explicitly instructed to do so and do not have access to the physical manipulatives themselves.

Methodologically, we will analyze this situation using a dialogical approach that scrutinizes how the pupils contextualize the task. This approach implies an interest in the resources that they rely on in making sense of the task, such as background knowledge, prior utterances or physical objects in their environment (Linell, 1998). In this sense, the context is not regarded as predetermined; it is instead the result of the participants' collaborative achievement in a particular activity. In line with this interest, the study draws on the notion of contextualization by Nilsson (2009), which has been specifically developed to analyze learning in mathematics. The specific research questions are as follows: How do the pupils contextualize the task given and how do they move between different contexts in their attempts to arrive at an answer? Our main interest is hence to explore in detail the phenomenon of pupils recalling and making use of manipulative-based equation-solving experiences when working with a task involving a different real world context. We will also discuss what support for and what obstacles to learning we can identify in the data and how these can inform practice.

2. Learning through manipulatives

In mathematics education literature, four common arguments exist for the use of manipulatives: they help pupils make sense of abstract ideas; they provide ways of testing and verifying ideas; they function as tools for problem solving; and they make mathematics learning more engaging (Burns, 2007). While concrete objects has obviously always been used in mathematics learning, the work of Dienes' (1969) gave rise to go in an increased use of concrete materials, such as the Dienes' blocks, that supposedly embodied mathematical structures. Dienes highlights the abstract understanding that can

[1] A concrete representation is defined as "the use of physical objects (e.g., manipulatives) or visual images (e.g., diagrams) to represent mathematical concepts and ideas and/or the conceptualization of abstract ideas in real-world situations (e.g., word problems)" (Ding & Li, 2014; p. 104)). An abstract representation is defined as the use of "symbols to represent mathematical concepts and ideas" (ibid.).