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Coordinating multiple representations of polynomials: What do patterns in students' solution strategies reveal?

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ABSTRACT

We investigate the strategies used by 64 advanced secondary mathematics students to identify whether a given pair of polynomial representations (graphs, tables, or equations) corresponded to the same function on an assessment of coordinating representations. Participants also completed assessments of domain-related knowledge and background skills. Cluster analysis of strategies by representation pair revealed patterns in the participants' strategy use. Two clusters were identifiable on tasks that required matching equations to graphs and graphs to tables. We identified overlap between these two clusters, suggesting that while the representation pair influenced strategy choice, there was also a general distinction between students who used more and less sophisticated strategies. However, students who used more sophisticated coordination strategies were similar to the others on measures of domain-specific knowledge or background skills. We consider implications for future investigations testing interventions to promote coordinating representations.

Problems in advanced secondary mathematics often require students to coordinate multiple external representations of functional relationships (Chang, Cromley, & Tran, 2016; Ferrara, Pratt, & Robutti, 2006; Janvier, Girardon, & Morand, 1993; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993). Teaching with multiple external representations can foster student understanding of important mathematical concepts and relationships (Brenner et al., 1997), and school mathematics policy documents recommend teaching with multiple representations (Department for Education, 2013; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The recommended focus on multiple representations builds upon decades of research by educational psychologists (Ainsworth, 2006; Mayer & Moreno, 2002; Rau, 2016) and mathematics educators (Acevedo Nistal, Dooren, Clarebout, Elen, & Verschaffel, 2009; Brenner et al., 1997; Hiebert & Carpenter, 1992; Leinhardt et al., 1990; Parnafes & Disessa, 2004; Yerushalmy, 1991) showing the benefit of multi-representational approaches. Accordingly, numerous teachers (e.g., Eichler & Erens, 2014) and curriculum authors (Chang et al., 2016) target skills such as constructing, coordinating, and reasoning with multiple representations as goals for instruction.

In order to design effective interventions to promote students' skills in coordinating multiple representations (CMR), researchers must identify related knowledge bases (Rau, 2016) and must also identify effective coordination strategies (Ainsworth, 2006). This study uses cluster analysis (Milligan & Hirtle, 2003) to identify profiles of CMR strategy use, and it extends prior research which has shown that students must have some domain-specific knowledge to coordinate representations in technical disciplines (Rau, 2016). We focused this initial work on representations of functions common in secondary mathematics: linear, quadratic, and cubic polynomials.

CMR strategies are actions used to identify whether two representations correspond to the same underlying function. For example, Fig. 1 shows a sample CMR task with an equation and graph. One CMR strategy is matching points on the graph with coordinates generated from the equation. Alternatively, a student might identify that the shape of the graph does not match the degree of the equation. Identifying and coding strategies is one way

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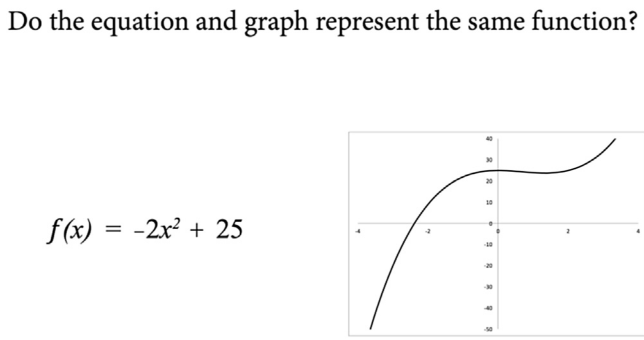

Fig. 1. Sample equation-graph item.

to begin understanding CMR skills. Analysis of strategy profiles helps identify patterns in strategy use. In what follows we present the results of a cluster analysis of the strategies used by 64 advanced secondary mathematics students to coordinate pairs of polynomial representations presented as items similar to Fig. 1. The results of this analysis suggest directions for future interventions designed to develop coordination skills.

1. Conceptual framework and prior research

Ainsworth's (2006) framework on multiple representations undergirds our focus on CMR, and Siegler's work on strategy selection (Siegler, 2005) motivated our decision to examine how students use strategies while solving the CMR tasks.

1.1. CMR and coordination strategies

Ainsworth's (2006) DeFT (Designs, Functions, Tasks) framework integrates research on teaching and learning with multiple external representations. Ainsworth, citing Yerushalmy (1991), described how teaching students to coordinate representations of functions in school mathematics is non-trivial. Rau's (2016) review suggests that efforts to teach CMR should account for both knowledge of individual students and socio-cultural characteristics of representation usage. Acevedo Nistal et al. (2009) argue that problem solving strategies and representational flexibility are connected to both the characteristics of the representations in use and the characteristics of the students interacting with the representations. This study builds on these frameworks through focusing on CMR task demands and knowledge. Less work in the area of CMR has explored how students deploy strategies to coordinate representations, and how these strategies are connected to characteristics of the representations that are considered. In order to explore profiles of strategy use in relation to the characteristics of representations we drew on Siegler's work on strategy selection.

Siegler's overlapping waves theory (2005) describes how, in general, learners use more sophisticated strategies across development. However, learners who can use a more sophisticated strategy may use a less sophisticated strategy on some problems, meaning strategy choice is not determined by level of development. For example, on the problem $3 + 8$, a child who has used the more sophisticated strategy of counting on from the larger summand may continue counting on from the smaller summand in some subsequent trials. Siegler notes that students choose strategies that “fit the demands of problems and circumstances and that yield desirable combinations of speed and accuracy, given the strategies and available knowledge that children possess” (Siegler, 2005, p. 771). Recent research has used Siegler's approach to analyze problem solving strategy choice among elementary and secondary mathematics students (Booth, Lange, Koedinger, & Newton, 2013; Jurdak & El Mouhayar, 2014), and we extend that work here in the area of CMR.

In contrast with Siegler's approach which studied the development of strategies to solve one type of problem, we use cluster analysis to *explore* profiles of strategy use for solving *different* types of problems. This exploratory work requires approaches like cluster analysis rather than variable-centered approaches like regression. This analytical technique, which is relatively rare in studies of strategy analysis, is described in more detail in the Methods section.

1.2. CMR strategies and views of function

Three of the most common function representations in school mathematics are graphs, tables, and equations. A review of literature on CMR skills identified that many secondary mathematics students struggle to coordinate representations with graphs (Chang et al., 2016; De Bock, van Dooren, & Verschaffel, 2015). As students learn to use and interpret graphs, one important development is transitioning from making point-by-point comparisons to more holistic comparisons of functions and graphs (Friel, Curcio, & Bright, 2001; Leinhardt et al., 1990; Yerushalmy, 1991). Making point-wise connections reflects a “process” view of a function, while global comparisons treat functions as “objects” (Moschkovich et al., 1993). Given nearly any representation pair, it is possible to identify whether two representations are the same function by matching ordered pairs (using a process view), but other strategies may yield accurate answers in less time. For example, as in Fig. 1 above, a student may answer more accurately and quickly by evaluating the global features of the function. Global features include the slope (for linear functions), direction, or degree (using an object view). Friel et al. (2001) suggest that comparing equations and graphs using a point-by-point method is a less sophisticated strategy than using global properties. However, this distinction may not necessarily apply to the coordination of tables and equations, where point-by-point comparisons are the only feasible option. In cases where point-by-point matching is the only option, however, point-wise CMR strategies may still vary by expertise. This highlights the importance of considering the representation pair in analyses of CMR strategy choice.

CMR strategies influence both problem solving speed and accuracy. That there is a tradeoff of speed and accuracy in problem solving is well documented (Wickelgren, 1977). Increasing speed generally decreases accuracy, while focusing on increased problem solving accuracy can slow performance. However, the speed-accuracy relationship is moderated by the strategy used to solve a problem. Some strategies allow problem solvers to increase accuracy while simultaneously maintaining or increasing increased speed. In this study, global comparisons of a function's shape or direction allow for relatively fast and accurate CMR. In contrast, point-by-point comparisons can be accurate, but time consuming, particularly when many values are calculated.

1.3. Summary

Prior research indicates that CMR strategy choice is related to both the level of student development in the domain and student characteristics such as background knowledge and skills. In the case of CMR, this study builds on prior work that has investigated connections between students' background skills, their domain-specific knowledge and their success coordinating multiple representations (Ainsworth, 2006; Cromley et al., 2017; Rau, 2016). We address the following research questions: