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Adaptive number knowledge and its relation to arithmetic and pre-algebra knowledge

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ABSTRACT

Traditionally measured skills with arithmetic are not related to later algebra success at levels that would be expected given the close conceptual relation between arithmetic and algebra. However, adaptivity with arithmetic may be one aspect of arithmetic competences that can account for additional variation in algebra attainment. With this in mind, the present study aims to present evidence for the existence and relevance of a newly acknowledged component of adaptivity with arithmetic, namely, adaptive number knowledge. In particular, we aim to examine whether there are substantial individual differences in adaptive number knowledge and to what extent these differences are related to arithmetic and pre-algebra skills and knowledge. Adaptive number knowledge is defined as the well-connected knowledge of numerical characteristics and relations. A large sample of 1065 Finnish late primary school students completed measures of adaptive number knowledge, arithmetic conceptual knowledge, and arithmetic fluency. Three months later they completed a measure of pre-algebra skills. Substantial individual differences in adaptive number knowledge were identified using latent profile analysis. The identified profiles were related to concurrent arithmetic skills and knowledge. As well, adaptive number knowledge was found to predict later pre-algebra skills, even after taking into account arithmetic conceptual knowledge and arithmetic fluency. These results suggest that adaptive number knowledge is a relevant component of mathematical development, and may help account for disparities in algebra development.

1. Introduction

In the transition from arithmetic to algebra, it is yet unclear exactly what types of basic skills are needed for later success. Traditional skills of calculation – the rapid algorithmic solving of typical problems – have been deemphasized in many circles of mathematics education. Indeed, it is not clear that basic whole-number arithmetic calculation skills (e.g. $5 + 12 = ?$) have a strong impact on later success with algebra when compared with other aspects of mathematical development such as rational number knowledge (Siegler et al., 2012), despite their logical connection.

Instead, many researchers of children's mathematical development have turned their concern to an alternate form of mathematical knowledge and skills. This new form sits in contrast to the static and calcified knowledge, with little transferability to new situations, referred to as *routine expertise*. Instead, a more malleable and interconnected set of knowledge and skills that is easily applied to new situations, referred to as *adaptive expertise,* is desired (Baroody, 2003; Hatano & Oura, 2003).

Within the frames of whole-number arithmetic, a core feature of adaptive expertise is adaptivity with arithmetic problem solving strategies. Adaptivity (which this study will focus on) refers to choosing and using the arithmetic problem solving strategy that is the most situationally-appropriate strategy for that person who is solving that particular problem (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Adaptivity with arithmetic has been linked with later success with mathematics, including algebra (Kieran, 1992). In particular, one cornerstone of adaptivity has been well-researched over the past twenty years, namely, the flexible

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switching between multiple strategies (e.g. Lemaire & Siegler, 1995; Torbeyns, Ghesquière, & Verschaffel, 2009). However, given that flexibility in terms of strategy choice may only make up a portion of what entails full adaptivity (Threlfall, 2009; Verschaffel et al., 2009), it is just as important to look more generally at what makes adaptivity with whole number arithmetic possible. This more general view of adaptivity with whole number arithmetic may also include the well-connected knowledge of numerical characteristics and relations, or adaptive number knowledge (McMullen et al., 2016). Thus, the present study aims to examine the nature of individual differences in adaptive number knowledge in late primary school students and how adaptive number knowledge is related to arithmetic and pre-algebra knowledge and skills.

1.1. Adaptive number knowledge

If adaptivity with arithmetic requires the choosing of the most appropriate strategy at the time, for that person, one must be able to flexibly switch between numerous problem solving strategies (Siegler & Lemaire, 1997). However, one must also be able to recognize the relevant numerical characteristics and relations within the problem that would suggest the most appropriate strategy. The well-connected network of numerical characteristics and relations that defines adaptive number knowledge is required for this recognition to happen (Threlfall, 2009). Previously, procedural flexibility has been suggested to be a necessary, but not sufficient, component of adaptivity (Threlfall, 2009; Verschaffel et al., 2009). Recently, McMullen et al. (2016) have argued that the advanced representation of numerical relations, which constitutes adaptive number knowledge, is a key requirement for adaptivity with arithmetic problem solving in varying situations.

The existence of a set of solution strategies to be drawn from when confronted by an arithmetic problem has been called into question (Threlfall, 2002, 2009). As an alternative, it is suggested that when students are faced with an arithmetic problem they formulate a solution strategy in-situ. Research in various domains of mathematical development suggest that there are large individual differences in the types of knowledge that students rely on, with some more reliant on procedural knowledge and others relying more on conceptual knowledge (Bempeni & Vamvakoussi, 2015; Hallett, Nunes, & Bryant, 2010). Within the realm of adaptivity with arithmetic, it is also possible that students rely on these different meta-strategies in different ways, with some students more reliant on choosing from a set of existing strategies, while others are actively using idiosyncratic strategies developed during the problem solving process.

With the in-situ creation of strategies, it is clear that one must first recognize the different relations and characteristics of the numbers that exist in the problem in order to determine the most effective solution strategy. Expert mathematicians were found to rely on "nice" numbers that had specific features and relations (such as the proximity of 59 to a multiple of 20) instead of standard algorithms when mentally solving arithmetic problems (Dowker, 1992). Likewise, even if these solution strategies come from an existing repertoire, as suggested by some research (Hickendorff, van Putten, Verhelst, & Heiser, 2010; Torbeyns, de Smedt, Ghesquière, & Verschaffel, 2009), the adaptive choice of the most appropriate strategy still requires recognizing the characteristics of the numbers present (e.g. numbers close each other across decades, like in 41–39). For example, there are larger individual differences in flexible strategy use with adaptive composition (e.g. $4 + 7 + 6 = 10 + 7 = 17$) than other strategies (Canobi, Reeve, & Pattison, 2003). One reason for this may be the need to focus on the numerical relations in order to recognize the opportunity to use this strategy. In general, the lack of connection between students' knowledge of potential solution strategies and their actual use (Blöte, Klein, & Beishuizen, 2000), especially when not explicitly guided to do so (Gaschler, Vaterrodt, Frensch, Eichler, & Haider, 2013), suggests that it is not sufficient to know about a strategy. Instead, students must be able to independently recognize when a particular strategy is appropriate, in relation to their own skills with that strategy, the socio-cultural context in which the problem exists, and the numerical features of that particular problem (Verschaffel et al., 2009).

1.2. Arithmetic knowledge and skills

There are two components of adaptive number knowledge that have been roughly defined previously, 'numerical knowledge and skills' and 'arithmetic calculation knowledge and skills' (McMullen et al., 2016). These closely connected features of mathematics learning inform the extent and interconnectedness of students' knowledge of numerical characteristics and relations, which describes their adaptive number knowledge.

The numerical knowledge and skills that impact a students' adaptive number knowledge include knowledge of the natural number system, including its base-ten structure. This includes the ability to recognize important and useful arithmetical relations between numbers and determine the key characteristics of numbers that would be useful for problem solving (Geary, Hoard, Byrd-Craven, & DeSoto, 2004). For example, these relations and characteristics include the amount of factors a number has (e.g. 24 has more factors than 25), its proximity to other useful or "nice" numbers (e.g. 123 is close to 11^2), and estimates of multiples of the number. In particular, precision in magnitude representation and estimation is expected to play a key role in adaptive number knowledge (Gallistel & Gelman, 1992; Siegler & Lortie-Forgues, 2014). In general, such skills are closely related to advanced number sense, as defined in much work on the later development of mathematical competences (Mou et al., 2016). Adaptive number knowledge expands this notion by integrating numerical skills and knowledge with calculation skills and knowledge.

The relation between adaptive number knowledge and students' arithmetic calculation knowledge and skills has been tentatively established in a previous study (McMullen et al., 2016). In particular, ninth graders' arithmetic fluency and conceptual knowledge have been found to be related to their adaptive number knowledge. Arithmetic fluency reflects the ability to rapidly work with the four arithmetic operations in a conventional form, and can be seen as a procedural fluency that mainly requires recall or algorithmic solution strategies (Rittle-Johnson, Siegler, & Alibali, 2001; Schneider, Rittle-Johnson, & Star, 2011). This ability to rapidly and accurately complete one-step arithmetic problems informs adaptive number knowledge by allowing for the quick assessment of potential arithmetic relations between numbers, for example in recognizing useful multiples (e.g. 12 and 3 are related through $3 \times 4 = 12$). Arithmetic conceptual knowledge – including knowledge of the order of operations, the commutativity principle, and the associativity principle – has also been shown to be related to adaptive number knowledge (McMullen et al., 2016). In order to have strong adaptive number knowledge, students must know the allowances and constraints of how numbers and operations can be used in arithmetic. This is particularly true with more complex arithmetic relations, such as using both additive and multiplicative operations to relate numbers (e.g. $30 \bullet 2 - 1 = 59$).

1.3. Adaptive number knowledge and pre-algebra skills

Previous research suggests that there is a relation between students' adaptive number knowledge and their arithmetic fluency