Learning and Instruction 47 (2017) 65–79

Contents lists available at ScienceDirect

Learning and Instruction

journal homepage: www.elsevier.com/locate/learninstruc

Children's understanding of the commutativity and complement principles: A latent profile analysis

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A R T I C L E I N F O

Article history:
Received 27 January 2016
Received in revised form
11 September 2016
Accepted 21 October 2016
Available online 2 November 2016

Keywords:
Additive reasoning
Commutativity principle
Complement principle
Latent profile analysis

A B S T R A C T

This study examined patterns of individual differences in the acquisition of the knowledge of the commutativity and complement principles in 115 five-to six-year-old children and explored the role of concrete materials in helping children understand the prinicples. On the basis of latent profile analysis, four groups of children were identified: The first group succeeded in commutativity tasks with concrete materials but in no other tasks; the second succeeded in commutativity tasks in both concrete and abstract conditions, but not in complement tasks; the third group succeeded in all commutativity tasks and in complement tasks with concrete materials, and the final group succeeded in all the tasks. The four groups of children suggest a developmental trend – (1) Knowledge of the commutativity and of the complement principles seems to develop from thinking in the context of specific quantities to thinking about more abstract symbols; (2) There may be an order of understanding of the principles – from the commutativity to the complement principle; (3) Children may acquire the knowledge of the commutativity principle in the more abstract tasks before they start to acquire the knowledge of the complement principle. This study contributes to the literature by showing that assessing additive reasoning in different ways and identifying profiles with classification analyses may be useful for educators to understand more about the developmental stage where each child is placed. It appears that a more fine-grained assessment of additive reasoning can be achieved by incorporating both concrete materials and relatively abstract symbols in the assessment.

1. Introduction

Children's understanding of logical principles in mathematics has received increasing empirical attention in recent years because of its importance in mathematical problem solving and computation (Baroody, Torbeyns, & Verschaffel, 2009; Ching & Nunes, 2016; Nunes, Bryant, Barros, & Sylva, 2012; Nunes et al., 2007; Verschaffel, Bryant, & Torbeyns, 2012). It has been argued that children may initially understand mathematical principles in the context of concrete referents (Bruner, 1960; Bryant, Christie, & Rendu, 1999; Gilmore & Papadatou-Pastou, 2009; Hughes, 1981; Piaget & Inhelder, 1975; Resnick, 1992; Vygotsky, 1962). Mathematics models various aspects of the world effectively by creating abstract structures that have properties shared with its real-world counterpart. We can manipulate and use the mathematical model to predict and make conclusions about events if the model acts in ways that truly corresponds to it. Some researchers have proposed that concrete materials can be used as an intermediary between the symbolic-mathematical world and the real world (Bruner, 1966; Piaget, 1952; Resnick, 1992). The concrete model is often considered more abstract than the actual situation, but less abstract than the mathematical model represented by numerical symbols. Thus, they may act as a vehicle through which children model the quantitative aspects of the real world. However, the notion that children's thinking is inherently concrete in nature is not universally accepted (Gelman & Wellman, 1991; Gelman, 2000, 2003). Some evidence suggests that concrete materials may facilicate the understanding of certain prinicples only (Canobi, Reeve, & Pattison, 2003). In the present study, we examined patterns of individual differences in children's knowledge of two essential prinicples in, additive reasoning, namely the commutativity principle and the complement principle. Using a 'person-centered' approach (Bergman & Magnusson, 1997; Bisanz, Watchorn, Piatt, & Sherman, 2009; Laursen & Hoff, 2006), we aimed to explore patterns of individual differences in children's performance on different

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http://dx.doi.org/10.1016/j.learninstruc.2016.10.008
0959-4752/

reasoning tasks (commutativity and complement principles) in different testing contexts (with and without the support of concrete materials).

1.1. The importance of understanding the commutativity and complement principles

Additive reasoning is based on quantities connected by part-whole relations. Two essential properties of part-whole relations are (1) the commutativity principle and (2) the complement principle (Kilpatrick, Swafoord, & Findell, 2001). Commutativity refers to the irrelevance of addend order to the sum, i.e. '$a + b = c$' implies '$b + a = c$', whereas the complement principle refers to the inverse relation between addition and subtraction, i.e. '$a + b = c$' implies '$c - a = b$'. These two prinicples are important for children to learn mathematics because they contribute to (1) the understanding of the nature of number, (2) computational fluency, and (3) the ablity to solve problems in a variety of situations.

Piaget (1952) argues that numbers are not simply a set of words in a fixed order, but they also reflect the part-whole logic of the number system. For example, the mastery of additive reasoning involves the integration of the commutativity and complement prinicples. One should understand that three quantities e.g., $3 + 4 = 7$ can be expressed in four mathematical relations, e.g., $7-3 = 4$, $4 + 3 = 7$, $7-4 = 3$, and $3 + 4 = 7$, and that these four expressions can be deduced from each other. A thorough understanding of the part-whole relations of quantities involves the recognition that these expressions are essentially describing the same relation.

The conceptual understanding of these two prinicples may contribute to children's computational fluency (Baroody et al., 2009; Canobi, 2004; Canobi et al., 2003; Nunes, Bryant, Hallett, Bell, & Evans, 2009). It has been suggested that this understanding may form the basis for children to develop more advanced computational strategies that help them modify complex problems to make them easier to solve (Canobi, 2004; Canobi et al., 2003; Fuson, 1990; Nunes & Bryant, 1996, 2015). For example, some efficient strategies (Gaschler, Vatterodt, Frensch, Eichler, & Haider, 2013; Shrager & Siegler, 1998), such as counting-all starting with the larger addend (CAL) and counting-on from the larger addend (COL), require the knowledge that the order of numbers does not affect the outcome in addition (i.e. the commutativity principle). The understanding of the commutativity principle may also foster the development of other strategies, such as the 'ten-strategy' and 'addends-compare strategy'. For example, children who grasp the commutativity principle can transform the problem '$3 + 6 + 7$' into '$(3 + 7) + 6$' that is easier to solve (the ten-strategy). For some arithmetic problems, children do not need to calculate if they recognise that the identical addends that had been shown (though in different order e.g., '$2 + 7 + 8$') in a previous problem had already been solved e.g., '$8 + 7 + 2$'. This addends-compare strategy may also require the understanding of the commutativity principle (Gaschler et al., 2013). An understanding of the inversion principle may also facilitate the use of 'indirect addition' in which children can use additions to solve subtraction problems effectively if the numbers are close to each other. For example, to solve '$21-18$', it is less likely to make mistakes if they count up from 18 to 21. Some researchers have suggested that the complement principle contributes to the mastery of basic subtraction combinations (Baroody, 1983, 1984, 1985, 1999; Baroody & Ginsburg, 1986; Baroody, Ginsburg, & Waxman, 1983; Fuson, 1988, 1992; Putnam, deBettencourt, & Leinhardt, 1990).

Understanding the commutativity and complement prinicples may also help children solve problems in a variety of situations. The solution to many story problems relies on the knowledge of the underlying relations between the quantities in the problem. Sometimes the relations are not obvious to problem solvers, especially when those problems whose solutions rest on the understanding of the inverse relation between addition and subtraction. For instance, children may not find a Change problem difficult when the missing information is the result of the change (e.g., 'David had 8 books. Then Peter gave him 3 more books. How many books does David have now?'). It is because the action in the story and the arithmetic operation required to solve the problem are consistent – A problem that involves a change that increases the quantitiy can be solved by addition, whereas one that decreases the quantity can be solved by subtraction. In contrast, when the starting situation is not known (e.g., 'Alex had some cookies. He gave 3 cookies to his mother and had 8 cookies left. How many cookies did he have before?'), problem solvers have to decide which arithmetic operation to use based on the information about the change and its end result. These start-unknown problems are more difficult (e.g., Carpenter, Hiebert, & Moser, 1981; De Corte & Verschaffel, 1987; Ginsburg, 1982) because the relation between the action described in the story and the operation is inverse, i.e., A problem that involves a change that decreases the quantity has to be solved by addition. Students must understand that the operation 'addition' can be conceived as the inverse of 'subtraction' and analyse the quantitative relations underlying the problem situation.

Knowledge of the commutativity principle may also relate to children's solving some missing addend problems (Nunes & Bryant, 2015). Consider this example 'Jane had 3 cookies, got some more and now has 7. How many more cookies did she get?' Children can easily solve this problem by representing the first addend with 3 fingers, counting up to the final state i.e. 7 fingers, and evaluated how many fingers they had to add in the process. However, if the problem has the first rather than the second addend missing e.g., 'Jane had some cookies; her mother gave her 4 more and now she has 7; how many did she have to start with?' the children have to understand that the order does not affect the total. Those who understand the commutativity principle can start from the second addend i.e. 4, add up to 7, and count how many were added. Children who do not understand commutativity may find this problem difficult to solve because they do not know how many cookies Jane to start with.

1.2. Using concrete materials to facilitate understanding

Given the importance of understanding the commutativity and complement principles in mathematics learning, we should identify ways to help children learn these principles. Some theorists and evidence suggest that young children can obtain cognitive benefits from exploring mathematical concepts with concrete materials (Bruner, 1960; Bryant et al., 1999; Gilmore & Papadatou-Pastou, 2009; Hughes, 1981; Piaget & Inhelder, 1975; Resnick, 1992; Vygotsky, 1962). Classic developmental theories contend that the acquisition of symbolic competence proceeds through a concrete-to-abstract shift: The progression from thinking that is based on concrete reality to thinking that is less constrained by context. For example, Piaget (1952) postulates that the development of the ability to reason with abstract hypothetical propositions without the help of more concrete information was the final stage of cognitive development. Piaget observed that children at the concrete operational stage had difficulty in reasoning about false propositions that included relations that could not happen in the real world.

Other popular theories have also seen development in terms of a transition from concrete to abstract. For example, in research of early categorization, Bruner (1966) argues that conceptual development is a perceptual-to-conceptual shift. At first, children can