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Excluding features in fuzzy relational compositions☆

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The aim of this paper is, first, to recall fuzzy relational compositions (products) and, to introduce an idea, how excluding features could be incorporated into the theoretical background. Apart from definitions, we provide readers with a theoretical investigation. This investigation addresses two natural questions. Firstly, under which conditions (in which underlying algebraic structures) the given three natural approaches to the incorporation of excluding symptoms coincide. And secondly, under which conditions, the proposed incorporation of excluding features preserves the same natural and desirable properties similar to those preserved by fuzzy relational compositions.

The positive impact of the incorporation on reducing the suspicions provided by the basic "circlet" composition without losing the possibly correct suspicion is demonstrated on a real taxonomic identification (classification) of Odonata. Here, we demonstrate how the proposed concept may eliminate the weaknesses provided by the classical fuzzy relational compositions and, at the same time, compete with powerful machine learning methods. The aim of the demonstration is not to show that proposed concept outperforms classical approaches, but to show, that its potential is strong enough in order to complete them or in order to be combined with them and to use its different nature.

1. Introduction

Fuzzy relational compositions (also called "products") possess an important role in distinct areas of fuzzy mathematics and their applications. Let us recall, e.g., the formal constructions of fuzzy inference systems (Pedrycz, 1985; Štěpnička & Jayaram, 2010) and related systems of fuzzy relational equations (De Baets, 2000; Di Nola, Sessa, Pedrycz, & Sanchez, 1989; Pedrycz, 1983; Sanchez, 1976), medical diagnosis (Bandler & Kohout, 1980b), architectures of information processing (Bandler & Kohout, 1985) or in flexible queries in relational databases (Dubois & Prade, 1996). They provide an extension of classical relational compositions and have been exhaustively studied by Willis Bandler and Ladislav Kohout in the late 70's and the early 80's. There are numerous scientific works deeply elaborating various aspects of the fuzzy relation compositions, see e.g. Belohlavek (2012); Běhounek and Daňková (2009); De Baets and Kerre (1993).

Although it has been a long time since the first developments, the topic is still very up-to-date and deserves an attention. This fact may be demonstrated by numerous recent works, focusing, for example, on inference systems (Lim & Chan, 2015; Mandal & Jayaram, 2015; Štěpnička, De Baets, & Nosková, 2010), modeling monotone fuzzy rule bases (Štěpnička & De Baets, 2013), flexible query answering systems (Pivert & Bosc, 2012) or on wireless communication (Yang, Zhou, & Cao, 2016). Another interesting direction of the research is the employment of generalized quantifiers (Cao & Štěpnička, 2015; Štěpnička & Holčapek, 2014) which was motivated again by the medical diagnosis and generally by the classification problem but its use is much wider, as it can be demonstrated by numerous works on flexible query answering systems and fuzzy relational databases (Delgado, Sanchez, & Vila, 2000; Pivert & Bosc, 2012) where the softening of the quantifier has been used already for years.

Thinking about the medical diagnosis problem or the classification problem in general, we show, how the excluding features, or excluding symptoms in the case of the diagnosis, may be incorporated into the fuzzy relational products in order to improve relevance and the precision of the suspicion provided the standard fuzzy relational products. The contribution of this extension will be demonstrated on an illustrative classification example.

The paper is organized as follows: In Section 2, we provide readers with preliminaries on the algebraic structures and their properties and we recall basic facts about relational and fuzzy relational compositions. In Section 3, we introduce the idea of excluding features and show how it can be employed in the fuzzy relational compositions. Furthermore, we present theoretical inves-

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tigation showing, under which conditions three natural definitions of the employment of excluding features coincide. Section 4 follows some of the previous studies (c.f. Běhounek & Daňková, 2009; De Baets & Kerre, 1993) presents the list of preserved properties for the newly defined concept. Finally, Section 5.1 provides readers with a demonstrative example and Section 6 with concluding remarks.

2. Preliminaries

2.1. Background algebraic structure of truth values

Let us recall the basic definitions of the used underlying algebraic structures. Let us also recall some properties preserved in these structures and that will be used in the sequel.

Definition 1. An algebra $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ is a *residuated lattice* if

1. $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least and the greatest element
2. $\langle L, \otimes, 0, 1\rangle$ is a commutative monoid such that $\otimes$ is isotone in both arguments
3. the operation $\to$ is a residuation with respect to $\otimes$, i.e.

$$a \otimes b \le c \quad \text{iff} \quad a \to c \ge b\,. \tag{1}$$

Let us list some of the useful properties that are immediately available to us for any $a, b, c \in \mathcal{L}$ (c.f. Novák, Perfilieva, & Močkoř, 1999):

$$a \to b = 1 \text{ whenever } a \le b\,, \tag{2}$$

$$a \to c \ge b \to c \text{ whenever } a \le b\,, \tag{3}$$

$$a \to b \le a \to c \text{ whenever } b \le c\,, \tag{4}$$

$$a \otimes b \le a,\ \ a \otimes b \le b, a \otimes b \le a \wedge b\,, \tag{5}$$

$$a \otimes (a \to b) \le b\,, \tag{6}$$

$$a \to (b \to c) = (a \otimes b) \to c = (b \otimes a) \to c\,, \tag{7}$$

$$(a \wedge b) \otimes c \le (a \otimes c) \wedge (b \otimes c), \tag{8}$$

$$(a \vee b) \otimes c = (a \otimes c) \vee (b \otimes c), \tag{9}$$

$$(a \vee b) \to c = (a \to c) \wedge (b \to c). \tag{10}$$

We can define additional operations for all $a, b \in L$, namely: biresiduation (biimplication, residual equivalence), negation, and addition, respectively:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a),$$
$$\neg a = a \to 0,$$
$$a \oplus b = \neg(\neg a \otimes \neg b).$$

Lemma 2. *Let $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ be a residuated lattice. Then for all $a, b \in L$:*

$$\neg(a \otimes b) \ge \neg a \oplus \neg b. \tag{11}$$

Proof. Due to (5) and the adjunction property, we get $a \le (a \to 0) \to 0$. Then using the definition of the addition, negation and the antitonicity of residual implication $\to$ in the first argument (3), we get

$$\neg a \oplus \neg b = (((a \to 0) \to 0) \otimes ((b \to 0) \to 0)) \to 0$$
$$\le (a \otimes b) \to 0 = \neg(a \otimes b).$$

□

Let us recall two more notions, that will be used in the sequel in order to narrow the class of residuated lattices where some desirable properties will be preserved.

Definition 3. Let $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ be a residuated lattice. We say that the negation $\neg$ is *strict*, if the following holds:

$$\neg a = \begin{cases} 0, & a > 0 \\ 1, & a = 0. \end{cases}$$

Furthermore, we say that an element $a \in L\backslash\{0, 1\}$ is a *zero divisor* of $\otimes$ if there exists some $b \in L\backslash\{0, 1\}$ such that $a \otimes b = 0$.

Finally, let us recall the MV-algebra.

Definition 4. An MV-algebra is an algebra $\mathcal{L} = \langle L, \oplus, \otimes, \neg, 0, 1\rangle$ with two binary operations $\oplus$, $\otimes$, a unary operation $\neg$ and two constants such that $\langle L, \oplus, 0\rangle$ and $\langle L, \otimes, 1\rangle$ are commutative monoids and the following identities hold:

$$a \oplus \neg a = 1, \qquad a \otimes \neg a = 0,$$
$$\neg(a \oplus b) = \neg a \otimes \neg b, \qquad \neg(a \otimes b) = \neg a \oplus \neg b,$$
$$a = \neg\neg a, \qquad \neg 0 = 1,$$
$$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a.$$

It is well known (c.f. Novák et al., 1999) that every MV-algebra $\langle L, \oplus, \otimes, \neg, 0, 1\rangle$ is a residuated lattice $\langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ by putting

$$a \vee b = \neg(\neg a \oplus b) \oplus b = (a \otimes \neg b) \oplus b,$$
$$a \wedge b = \neg(\neg a \vee \neg b) = (a \oplus \neg b) \otimes b,$$
$$a \to b = \neg a \oplus b,$$

where $\to$ is a residuation operation with respect to $\otimes$. An MV-algebra or residuated lattice is complete if the underlying lattice is complete.

Furthermore, let us recall one more algebraic structure, in particular, the Heyting algebra. The Heyting algebra is a residuated lattice with $\otimes = \wedge$, i.e., $\langle L, \vee, \wedge, \to, 0, 1\rangle$.

Let us recall, that in a complete residuated lattice, the following holds for any $a, b, c \in \mathcal{L}$ and for any index set I:

$$\bigvee_{i \in \mathcal{I}} a_i \to b = \bigwedge_{i \in \mathcal{I}} (a_i \to b)\,. \tag{12}$$

As a consequence of (12) and the definition of the negation, we immediately get the following equality:

$$\neg \bigvee_{i \in I} a_i = \bigwedge_{i \in I} \neg a_i.$$

The well-known examples of residuated lattices are the Łukasiewicz algebra, the Gödel algebra or the Goguen algebra. The two latter ones, however, are not MV-algebras and thus, the double negation law does not hold in these algebras. For details, we refer interested readers to the relevant literature (Hájek, 1998; Novák et al., 1999).

2.2. Relational and fuzzy relational compositions

Let us recall some basic facts about relational compositions, from which the fuzzy relational compositions naturally stem.