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Retrieving sinusoids from nonuniformly sampled data using recursive formulations

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ABSTRACT

A heuristic procedure based on novel recursive formulation of sinusoid (RFS) and on regression with predictive least-squares (LS) enables to decompose both uniformly and nonuniformly sampled 1-d signals into a sparse set of sinusoids (SSS). An optimal SSS is found by Levenberg–Marquardt (LM) optimization of RFS parameters of near-optimal sinusoids combined with common criteria for the estimation of the number of sinusoids embedded in noise. The procedure estimates both the cardinality and the parameters of SSS. The proposed algorithm enables to identify the RFS parameters of a sinusoid from a data sequence containing only a fraction of its cycle. In extreme cases when the frequency of a sinusoid approaches zero the algorithm is able to detect a linear trend in data. Also, an irregular sampling pattern enables the algorithm to correctly reconstruct the under-sampled sinusoid. Parsimonious nature of the obtaining models opens the possibilities of using the proposed method in machine learning and in expert and intelligent systems needing analysis and simple representation of 1-d signals. The properties of the proposed algorithm are evaluated on examples of irregularly sampled artificial signals in noise and are compared with high accuracy frequency estimation algorithms based on linear prediction (LP) approach, particularly with respect to Cramer–Rao Bound (CRB).

1. Introduction

1.1. Problem statement

Let $\{w_k\}_{k=1}^{K}$ denote a time series, where $w_k \in \Re (k = 1, \ldots, K)$ is the kth observation obtained at the corresponding time point t_k, $\{t_k\}_{k=1}^{K}$. Suppose a time series representing a finite number of sine waves embedded in noise. Suppose also that a time series may have a nonzero mean value and/or a linear trend. The objective of this paper is spectral analysis and modeling of a time series outlined above and represented by:

$$w_k = o + \kappa t_k + \sum_{n=1}^{N} [A_n \cdot \sin(\omega_n \cdot t_k + \varphi_n)] + s_k, k = 1, \ldots, K, \quad (1)$$

where o and κ denote the corresponding y-intercept at $t = 0$ and the slope of a linear trend line, A_n, ω_n and φ_n are the corresponding amplitude, radian frequency and phase of the nth sine wave and s_k represents the noise.

1.2. Related work

A non-uniform sampling is common to many long-time ground-based astronomical observations including spectra and time series (Lomb, 1976; Scargle, 1982). A number of papers dealing with the decomposition of a time series into a SSS are based on the least-squares spectral analysis and have been published very early. Methods based on the least-squares fit of sinusoids to data are introduced, also known as LS periodogram (LSP) analysis, formulated as LS fitting problem:

$$\min_{\substack{A_n \geq 0 \\ \omega_n \in [0, \omega_{\max}] \\ \varphi_n \in [-\pi, \pi]}} \sum_{k=1}^{K} \left[w_k - \sum_{n=1}^{N} A_n \cdot \sin(\omega_n \cdot t_k + \phi_n) \right]^2, \quad (2)$$

where ω_{max} denotes maximum expected angular frequency. Frequency estimation methods can be divided into the two main classes: nonparametric and parametric. The nonparametric frequency estimation is based on the Fourier transform and its ability to resolve closely spaced sinusoids is limited by the length of sampled data. On the other hand the parametric approach enables to achieve a higher resolution since it assumes the generating model with known functional form, which satisfies the signal (So, Chan, Chan, & Ho, 2005).

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The earliest nonparametric frequency estimation methods are based on LSP analysis. Barning (1962) used least-squares fitting to calculate the amplitudes of sine waves from the corresponding frequencies selected from periodogram. Vaníček (1969) first proposed successive spectral analysis of equally spaced data and later he extended the analysis to nonumiformly sampled data (Vaníček, 1970). Lomb, (1976) analyzed statistical properties of irregularly spaced data based on periodogram analysis. He has shown that, due to the correlation between noise at different frequencies, noise has less effect on a spectrum than it could be expected. Scargle (1982) studied the use of periodogram with irregularly spaced data. He concluded that periodogram analysis and least-squares fitting of sine waves to data are exactly equivalent. Foster (1995) proposed a sequential method for removing false peaks from power spectra that can be viewed as Matching Pursuit (Mallat & Zhang, 1993), a general procedure for computing adaptive signal representations which decomposes any signal into a linear expansion of waveforms that are selected from a redundant dictionary of functions. Bourguignon, Carfantan, and Idier, (2007) estimated spectral components from irregularly sampled data. Sparse representation of noisy data is searched for in an arbitrarily large dictionary of complex-valued sinusoidal signals, which can be viewed as Basis Pursuit Denoising problem (Chen, Donoho, & Saunders, 2001). The nonparametric method for spectral analysis of nonuniform sequences of real-valued data named real-valued iterative adaptive approach (RIAA) is proposed by Stoica, Li, and He (2009). It can be interpreted as an iteratively weighted LSP. The method can be used for spectral analysis of general data sequences but is most suitable for zero-mean sequences with discrete spectra. Similar problems, dealing with sparse reconstruction, have been investigated recently in scope of compressed sensing, (Boufounos, Cevher, Gilbert, Li, & Strauss, 2012; Nichols, Oh, & Willett, 2014; Panahi & Viberg, 2014; Tang, Bhaskar, Shah, & Recht, 2012; Teke, Gurbuz, & Arikan, 2013), illustrating only signal reconstruction errors but not demonstrating that the proposed methods achieve a Cramer–Rao bound, above some SNR threshold, for all the real frequencies embedded in the signal.

Well-known parametric frequency estimation methods are maximum likelihood (ML) (Bresler & Macovski, 1986; Rife, & Boorstyn, 1976), and nonlinear least squares (NLS) (Stoica & Nehorai, 1988) and the methods based on linear prediction (LP) property of sinusoids like Yule–Walker equations (Chan, & Langford, 1982), total least squares, (Rahman, & Yu, 1987), iterative filtering (Li, & Kedem, 1994), MUSIC and ESPRIT (Porat, 2008), weighted least squares (So et al., 2005). Under additive white Gaussian noise the ML and NLS methods are equivalent and achieve Cramer–Rao lower bound (CRLB) asymptotically, but they are computationally demanding. The above mentioned methods, based on LP property, provide suboptimum estimation performance but they are computationally efficient. The parametric methods based on linear prediction (Chan, Lavoie, & Plant, 1981; Dash, & Hasan, 2011; So et al., 2005; Yang, Xi, & Guo, 2007) enable to retrieve the sinusoids from a uniformly sampled sinusoidal signal in noise when the number of sinusoids in the signal is known *a priori*. So et al. (2005) developed two high accuracy frequency estimators for multiple real sinusoids in white noise based on the LP approach. First, they developed a constrained least squares frequency estimator named reformulated Pisarenko harmonic decomposer (RPHD) and then they improved it through the technique of weighted least squares (WLS) with a generalized unit-norm (WLSun) and monic (WLSm) constraint. The method assumes uniformly sampled data and the number of sinusoids to be known *a priori*.

The heuristic procedure elaborated in this paper is also based on the LP property of a sinusoid and is intended for recovery of frequency-sparse signals in noise. It can be used in signal processing, machine learning and expert and intelligent systems to facilitate solving the classification, diagnosis, monitoring or process control tasks needing analysis and parsimonious representation of signals, including the signals in technical systems, bio-signals, astronomical observations, etc. The proposed algorithm enables to retrieve the sinusoids from either uniformly or nonuniformly sampled data. In order to adapt it to nonuniform sampling we first reformulate the LP property of a sinusoid and we named it a recursive formulation of a sinusoid (RFS). Then we formulate a sinusoidal model based on RFS and the corresponding procedures for the estimation of RFS parameters based on the minimization of LS error. By combining the RFS approach with the well-known methods for the estimation of the number of sinusoids in noise the proposed procedure enables to retrieve the sinusoids iteratively, one at a time, and to determine the order of the generating model. The proposed method assumes neither a zero mean sequence nor the number of sinusoids in a signal to be known *a priori*. The accuracy of the frequency estimation procedure proposed in this paper is compared with very high accuracy of frequency estimation obtained by LP approach reported by So et al. (2005). For a frequency-sparse signal the computational complexity of both methods is comparable, $O(K^3)$.

1.3. Methods for detection of the number of sinusoids

Most parametric methods for detection of sinusoids corrupted with noise minimize the sum of a data fit (likelihood) term and the complexity penalty term where the penalty term is usually derived via Akaike information criterion (AIC) (Akaike, 1974), Bayesian information criteria (BIC) (Schwarz, 1978) or minimum description length (MDL) (Rissanen, 1978). A review of information criterion rules for model-order selection with the summary of necessary steps used to adapt a rule to a specific problem is given in Stoica and Selen (2004). In this paper the attention is restricted to efficient detection criteria (EDC) type estimators (Djurić, 1996, 1998; Nadler & Kontorovich, 2011). EDC type estimators determine the number of sinusoids by minimizing:

$$\widehat{M} = \arg \min_{M=0,1,2\ldots} -\ln L\left(\widehat{\beta}_M, \mathbf{w}\right) + MC_K, \tag{3}$$

where $\mathbf{w}$ is the observed time series of length K, $\widehat{\beta}_M$ are parameter estimates of a model of order M, $L(\widehat{\beta}_M, \mathbf{w})$ is the corresponding likelihood term and C_K is the model-complexity penalty term that captures the dependency of the penalty on the number of samples K. For the unknown noise level the log-likelihood term in (3) can be approximated by:

$$\ln L\left(\widehat{\beta}_M, \mathbf{w}\right) = -\frac{K}{2} \ln \left\{ \sum_{k=1}^{K} \left[w_k - P_{M,k}\left(\widehat{\beta}_M\right)\right]^2 \right\}, \tag{4}$$

where $P_{M,k}(\widehat{\beta}_M)$ denotes the approximation of w_k at time point t_k made by a model of order M. By substituting (4) for log-likelihood in (3) we obtain:

$$\widehat{M} = \arg \min_{M=0,1,2,\ldots} \frac{K}{2} \ln \left\{ \sum_{k=1}^{K} \left[w_k - P_{M,k}\left(\widehat{\beta}_M\right)\right]^2 \right\} + MC_K. \tag{5}$$

By considering a Bayesian formulation and selecting the model with maximum *a posteriori* probability (MAP) criterion for sinusoids with unknown frequencies amplitudes and phases Djurić (1996) derived the following penalty term

$$C_K = (5M/2) \ln K \tag{6}$$

and he concluded that the parameters that can be determined more precisely should receive stronger penalty. Nadler and Kontorovich (2011) proposed the estimator inspired by ideas from extreme value theory (EVT) and the maxima of stochastic fields with