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Extension of a class of decomposable measures via generalized pseudo-metrics ☆

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Abstract

This study considers the application of generalized pseudo-metrics to the extension of decomposable measures. We prove that the extension of a non-strict Archimedean t-conorm-based σ-decomposable measure can be formulated as the closure of a subset of a certain generalized pseudo-metric space. We show that the extension via generalized pseudo-metrics is equivalent to the completion of t-conorm-based σ-decomposable measures and the well-known Carathéodory extension.

Keywords: Decomposable measure; Generalized pseudo-metric space; T-conorm; T-norm

1. Introduction

In the past 40 years, a substantial theory of nonadditive measures (also called fuzzy measures) has been developed. One class of fuzzy measures comprises triangular conorm decomposable measures ($\perp$-decomposable measures) [6,25,48], which include the λ-additive measure, probability measure, and possibility measure [33,47] as special cases. Many elegant results have been obtained regarding $\perp$-decomposable measures and related integrals [1,4,13,17,19, 22–24,27,34,37,38,42,50]. Based on these measures and related integrals, pseudo-analysis [36] was developed as a generalization of the classical analysis. In addition, the tools of pseudo-analysis have many applications in various fields such as decision theory, optimization problems, nonlinear partial differential equations, hybrid utility, pattern recognition, and aggregation analysis [2,7,8,10,13,20,29,30,32,33,35,36,43,46].

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It is well known that probabilistic metric spaces [44] and fuzzy metric spaces [9,11,21] constructed via a triangular norm are generalizations of the classic metric spaces. The connections among $\perp$-decomposable measures, various probabilistic metrics, and fuzzy metrics have been studied previously [12,15,16,28,39–41,45]. It should be noted that Qiu et al. [41] developed an approach for constructing a KM fuzzy pseudo-metric from a given $\perp$-decomposable measure. Using the KM fuzzy pseudo-metric, a class of decomposable measures can be extended to a σ-algebra. Subsequently, we constructed a pseudo-metric (in the sense of Pap) based on the measurable sets of a given σ-$\perp$-decomposable measure, and we also analyzed the connection between the σ-$\perp$-decomposable measure and induced pseudo-metric [49]. In the present study, we apply the pseudo-metric constructed in [49] to the extension of σ-$\perp$-decomposable measures.

The approach employed for extending fuzzy measures is an interesting topic in the theory of fuzzy measures. However, as noted by Wang and Klir [47], although it is impossible to provide a unified extension theorem for all types of fuzzy measures, several extension theorems can be established for some special classes of fuzzy measures. The necessary and sufficient conditions for extending a submeasure (a special null-additive monotone set function) from a ring to the generated σ-ring were first obtained by Dobrakov [5]. Similarly, Pap [26] gave the necessary and sufficient conditions for the extension of $\perp$-decomposable measures to monotone order continuous $\perp$-subdecomposable set functions. Thus, a theorem based on extending null-additive set functions from a ring to the algebra generated by the ring was proved by Pap [31]. [41] asked whether a σ-$\perp$-decomposable measure on an algebra $\mathcal{A}$ can be extended to a unique σ-$\perp$-decomposable measure on the generated σ-algebra $S(\mathcal{A})$. As shown by [41], if μ is a (NSA)-type σ-$\perp$-decomposable measure, then by using Carathéodory's extension method, μ can be uniquely extended to the σ-algebra $S(\mathcal{A})$. It is rather difficult to intuitively understand Carathéodory's extension method [14], so we present a pseudo-metric approach for the extension of the decomposable measures. Our results suggest that extending a σ-$\perp$-decomposable measure from an algebra to a σ-algebra can be reduced to finding the closure of a subset of a generalized pseudo-metric space.

The remainder of this paper is organized as follows. In Section 2, we provide some basic notions and auxiliary results that are needed later. In Section 3, we prove that the extension of a (NSA)-type σ-$\perp$-decomposable measure can be formulated as the closure of a subset of a certain generalized pseudo-metric space. In Section 4, we discuss the completion of σ-$\perp$-decomposable measures, as well as establishing the connection between this completion and the particular extension via generalized pseudo-metrics. To illustrate the effectiveness of our method for extension via pseudo-metrics, we compare it with Carathéodory's extension method in Section 5. In particular, the extension via generalized pseudo-metrics is equivalent to the completion of σ-$\perp$-decomposable measures and the well-known Carathéodory extension.

2. Preliminaries

Throughout this study, the notations $\mathcal{A}$, $S(\mathcal{A})$, and $\mathcal{P}(X)$ denote an algebra of subsets of the given nonempty set X, the σ-algebra generated by $\mathcal{A}$, and the power set of X, respectively. First, we recall the concepts of a triangular norm and triangular conorm from [7,18,44,48].

Definition 2.1.

(i) A function $\top : [0,1]^2 \to [0,1]$ is called a *triangular norm* (*t-norm*) if it satisfies the following conditions for all $x, y, z \in [0,1]$:
   (T1) $x \top 1 = x$, (boundary condition)
   (T2) $x \top y \le x \top z$ whenever $y \le z$, (monotonicity)
   (T3) $x \top y = y \top x$, (commutativity)
   (T4) $x \top (y \top z) = (x \top y) \top z$. (associativity)

(ii) A function $\perp : [0,1]^2 \to [0,1]$ is called a *triangular conorm* (*t-conorm*) if for all $x, y, z \in [0,1]$ it satisfies (T2)–(T4) and
   (S1) $x \perp 0 = x$. (boundary condition)

(iii) For any t-conorm $\perp$, the t-norm $\perp^*$ defined by
   (TCO) $x \perp^* y = 1 - (1-x) \perp (1-y)$,
   is called the dual t-norm of $\perp$.

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