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Properties of extremal families of MN-convex (MN-concave) functions

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Abstract

In this paper MN-convex and MN-concave functions are examined and compared for particular cases of M and N. Characterization of pairs M, N for the family of MN-convex (MN-concave) functions consisting of all functions and partial characterization of pairs M, N for the family of MN-convex functions consisting of all constant functions will be presented. The condition of MN-convexity was proposed by Aumann in 1933. It was convexity with respect to arbitrary binary means M and N (abbreviated to MN-convexity). Recently many authors have considered this notion with suitable pairs of means. Here aggregation functions M, N will be used for this type of convexity and concavity.

Keywords: Convex function; Concave function; Mean; Aggregation function; MN-convexity; MN-concavity; Geometric convexity; Logarithmic convexity; Quasi-convexity

1. Introduction

The notion of convex functions was introduced in 1905 by Jensen [15]. Then the condition of convexity was generalized by many authors. For example, Hardy [11] introduced the condition of logarithmic convexity in 1915. In 1928, Montel [22] introduced the condition of geometric convexity (multiplicative convexity) and von Neumann [23] introduced the condition of quasi-convexity. Next, more general condition of convexity, i.e. MN-convexity and M-convexity with respect to arbitrary means M and N, was proposed in 1933 by Aumann [3].

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Recently many authors have dealt with these generalizations. In particular, Niculescu [24] compared MN-convexity with relative convexity. Andersen et al. [2] examined inequalities implied by MN-convexity. Matkowski [20] examined inclusions between classes of MM-convexity. Krassowska [17] examined MN-convexity with arbitrary binary operations M and N.

The goal of this paper is a comparison of families of MN-convex functions for particular cases of M and N, and under some relations between M and N. Moreover, characterizations of pairs M, N for the family of MN-convex (MN-concave) functions consisting of arbitrary functions or constant functions will be presented (here only partial characterization). This paper is a continuation of the works presented at AGOP 2105 conference [5,14].

The presented here results may be applied in geographical information systems (GIS). Work with surfaces is one of the main tasks in geographical information systems. The data describing an area may often be influenced by some noise, hence there are various tools in GIS software to handle such cases (see [13]).

Also, sometimes it is necessary to convert surface data into more usable information. In GIS terminology we speak of reclassifying the surface. Reclassifying replaces a range of values by a single value. This is exactly an example of using an aggregation function, in mathematical language. (Another GIS function that can be represented by aggregation, is the overlay procedure.) The reclassification may be done also in such a way that areas with cells above a given value, or between two critical values, are given one code, and other areas are given another. Reclassifying surfaces is often done to reduce the number of output categories for an overlay analysis.

The algorithms used in GIS software to replace a cell value in such cases is mostly based on computing the value based on values of adjacent cells. Thus, if $f(y)$ is the value to be computed and $f(x)$, $f(z)$ are the values of adjacent cells, then $f(y) = N(f(x), f(z))$, where N is an appropriate function.

Such computations lead directly to the study of mappings, for which the values inside the interval depends on some transformation of the endpoint values. Various types of software use various methods for reclassifying. The question of the range width for such methods can lead us to the following problem: how large is the set of all possible reclassified surfaces for a fixed method? Or, in other words – for which methods this range is maximal or minimal? From mathematical point of view we ask for extreme function families fulfilling given inequalities involving suitable aggregation functions.

The contribution is organized as follows. In Section 2 we refer to the most popular examples and families of aggregation functions. Next, we discuss elementary consequences of Aumann's definition in Section 3. In Sections 4 and 5 some inclusions between families of MN-convex functions are presented. Sections 6 and 7 contain characterizations of the greatest families of pairs M, N such that $C(M, N) = ALL$ and $C^*(M, N) = ALL$ and a partial characterization (for bounded intervals $[a, b]$) of the pairs M, N such that the family of MN-convex functions is the smallest one: $C(M, N) = CONST$ (cf. Definition 4). Finally, some open problems implied by the results presented are mentioned in Section 8.

2. Binary operations

In this section and throughout the paper we will consider binary operations only.

Definition 1. Let $D \subset \mathbb{R}$ where D is an interval. An aggregation function in D (cf. [10]) is a function $M : D^2 \to D$ which is increasing, i.e. for any $x_1, x_2, y_1, y_2 \in D$:

$$x_1 \leqslant y_1, \quad x_2 \leqslant y_2 \Rightarrow M(x_1, x_2) \leqslant M(y_1, y_2), \tag{1}$$

and fulfills boundary conditions

$$\inf_{x,y \in D} M(x, y) = \inf D, \quad \sup_{x,y \in D} M(x, y) = \sup D. \tag{2}$$

A function $M : D^2 \to D$ is called a mean in D if (cf. [8,20,25])

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y), \quad x, y \in D. \tag{3}$$

Definition 2 *(cf. [7,21])*. Let $M : D^2 \to D$. Function M is called:

- symmetric if $M(x, y) = M(y, x)$ for $x, y \in D$,
- locally internal in D if $M(x, y) \in \{x, y\}$ for $x, y \in D$.