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Steady states of max-Łukasiewicz fuzzy systems

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Abstract

The paper gives a systematic characterization of the eigenspaces in a max-t algebra, where t is the Łukasiewicz t-norm. A max-Łukasiewicz fuzzy algebra can be used for the description of the states of discrete-event systems. The states can represent a balance between the resources expended during the run of a system (for example fuel or money). The classification of max-Łukasiewicz eigenspaces is described and illustrated by two- and three-dimensional examples: in this case it is possible to accompany the example with graphs. However, the description of the eigenspaces for higher dimensions is also outlined.

Keywords: Fuzzy algebra; t-Norms; Max-Łukasiewicz algebra; Steady states

1. Introduction

Extremal algebras are used mainly for describing and studying systems working in discrete time. During the operation of such a system, a steady state can arise. The system is described by a transition matrix and the eigenvector(s) of that matrix represent the steady states of the system.

A max-t fuzzy algebra is defined over the interval [0, 1] and uses, instead of the conventional operations of addition and multiplication, the operations of maximum and one of the triangular norms, the so-called t-norms. These operations are extended in a natural way to the Cartesian products of vectors and matrices. T-norms were introduced by Schweizer and Sklar in [9] in the context of probabilistic metric spaces, and are used to interpret the conjunction in fuzzy logics and the intersection of fuzzy sets. These functions have applications in many areas, such as decision making, statistics, game theory, information and data fusion, probability theory, and risk management. The t-norms together with the t-conorms play a key role in fuzzy set theory.

The fuzzy extension of description logics (the formalism for the representation of structured knowledge used frequently in the design of ontologies) is described in [1]. The applications of ontologies have increased in the last decades. Ontologies have been successfully used as part of expert and multiagent systems, as a knowledge base in robotics, as well as the core element in the Semantic Web (which aims at converting the current Web into a "web of data" by defining the meaning of information).

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Although there exist various t-norms and families of t-norms (for an overview, see, e.g., [5,2]), let us mention the four main t-norms: the Łukasiewicz t-norm, the Gödel t-norm, the Product t-norm, and the Drastic t-norm.

The Łukasiewicz norm is computed as

$$x \otimes_L y = \max\{x + y - 1, 0\}. \tag{1}$$

The Gödel norm is the simplest norm: the conjunction is defined as the minimum of the entries: that is, of the truth degrees of the constituents. Gödel logic is a logic with a relative comparison.

$$x \otimes_G y = \min(x, y) \tag{2}$$

The definition of the product norm is

$$x \otimes_p y = x \cdot y. \tag{3}$$

The drastic triangular norm (in the literature, there can be also found the term "weakest norm") is the basic example of a non-divisible t-norm on any partially ordered set. The drastic triangular norm is defined as follows:

$$x \otimes_{\mathrm{d}} y = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{if } \max(x, y) < 1 \end{cases} \tag{4}$$

Eigenvalues and eigenvectors are an important characteristic of a system described by these fuzzy algebras. For the case of the drastic and product t-norms, the eigenspace structure has already been studied. The paper [4] published in the journal *Fuzzy Sets and Systems* deals with the max-drast algebra. The investigation of the eigenspace in a max-prod algebra [8] is in preparation. Finally, [7,3] deal with the Łukasiewicz fuzzy algebra.

The state of a system at time t can be described by a state vector, say $x(t)$. The transition matrix, denoted by A, describes the transitions of the system from one state to another. By the multiplication of the transition matrix with the state vector, the next state of the system, $x(t + 1)$, is obtained; it can be written $A \otimes x(t) = x(t + 1)$. During the operation of the system, after some time, it can happen that the system reaches a steady state. In a max-Łukasiewicz fuzzy algebra, the state vectors of the steady states correspond to the eigenvectors of the transition matrix A.

Łukasiewicz arithmetical conjunction can be used in many types of situations. The fact that the number 1 is subtracted from the sum of the components and the maximum with zero is taken, leads to the observation that the count of the operation is the remainder, some part that is over the unit. Following this idea, the conjunction can be used to compute, for example, the amount of money that should be paid off for the phone bill where x can be the total price for the SMSs and y can represent the total price for the calls. The number 1 here is substituted by the amount of the lump sum. There are other similar situations, for example, data backup on the server, the maximal capacity of a pond, a project cost overrun, or the savings of partners with a common bill.

2. Eigenproblem in max-Łukasiewicz fuzzy algebras

A max-Łukasiewicz algebra uses, as was written above, two binary operations: addition, $x \oplus y = \max(x, y)$, and multiplication, $x \otimes_L y = \max\{x + y - 1, 0\}$. The operation $\otimes_L$ has, as in conventional algebra, priority over the operation $\oplus$. These operations are formally extended to matrices and vectors, again, similarly to linear algebra. That is, if the matrices $A = (a_{ij})$ and $B = (b_{ij})$ are of compatible sizes, then one can write $A \oplus B = (a_{ij} \oplus b_{ij}) = \max(a_{ij}, b_{ij})$; $A \otimes_L B = \bigoplus_k (a_{ik} \otimes_L b_{kj}) = \max_k(\max(a_{ik} + b_{kj} - 1, 0))$; $\alpha \otimes_L A = A \otimes_L \alpha = (\alpha \otimes_L a_{ij})$ for $\alpha \in \mathcal{R}$.

To solve the eigenproblem in a max-Łukasiewicz algebra means to find a nontrivial vector x (called the eigenvector) such that for some λ (called the eigenvalue)

$$A \otimes_L x = \lambda \otimes_L x. \tag{5}$$

This equation can be converted into the language of the so-called tropical linear algebra, i.e., the max-plus algebra, because, as we will see, these two structures are closely related. The max-plus algebra is denoted by $\mathcal{R}_{\max} = (\overline{R}, \oplus, \otimes, \varepsilon, e)$, where $\overline{R}$ is the set of real numbers extended by the infinite value $\varepsilon = -\infty$ and the zero element $e = 0$. The binary operations $\oplus, \otimes$ are defined on $\overline{R}$ by $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$.

The developed theory of the well known max-plus algebra can be applied in the following way. Let's have a closer look at the Łukasiewicz conjunction and rewrite the equation using max-plus operators:

$$x \otimes_L y = (x - 1) \otimes y \oplus 0. \tag{6}$$

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