ARTICLE IN PRESS
FSS:7045

Available online at www.sciencedirect.com
ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

www.elsevier.com/locate/fss

Pointwise aggregation of maps: Its structural functional equation and some applications to social choice theory

L. De Miguel [a], M.J. Campión [b], J.C. Candeal [c], E. Induráin [b,∗], D. Paternain [a]

[a] *Institute of Smart Cities and Departamento de Automática y Computación, Universidad Pública de Navarra, 31006 Pamplona, Spain*
[b] *Instituto de Investigación InaMat and Departamento de Matemáticas, Universidad Pública de Navarra, 31006 Pamplona, Spain*
[c] *Departamento de Análisis Económico, Universidad de Zaragoza, 50009 Zaragoza, Spain*

Received 16 November 2015; received in revised form 2 May 2016; accepted 18 May 2016

Abstract

We study a structural functional equation that is directly related to the pointwise aggregation of a finite number of maps from a given nonempty set into another. First we establish links between pointwise aggregation and invariance properties. Then, paying attention to the particular case of aggregation operators of a finite number of real-valued functions, we characterize several special kinds of aggregation operators as strictly monotone modifications of projections. As a case study, we introduce a first approach of type-2 fuzzy sets via fusion operators. We develop some applications and possible uses related to the analysis of properties of social evaluation functionals in social choice, showing that those functionals can actually be described by using methods that derive from this setting.

Keywords: Aggregation operators; Functional equations; Pointwise aggregation; Real-valued functions; Type-2 fuzzy sets; Applications to social choice

1. Introduction and motivation

Given a finite collection of maps $\{f_1, f_2, \dots, f_n\}$ from a set X into another set Y, a new map $f_{n+1} : X \to Y$ obtained in some manner from the given maps $f_1, f_2, \dots, f_n$ is said to be an aggregation of those maps. A typical example is the arithmetic mean $f_{n+1} = \frac{f_1+\dots+f_n}{n}$ of n real-valued functions (here $Y = \mathbb{R}$, the real line).

Since information fusion appears in almost every application, aggregation has become a crucial technique and has generated a broad field of research (see, for example [8,9,15]). From an applied point of view, aggregation functions have also been used for solving real-world problems, for example in fuzzy set theory [3,13] or image processing [12, 38,26].

∗ Corresponding author. Tel.: +34 948169551; fax: +34 948166057.
E-mail address: steiner@unavarra.es (E. Induráin).

http://dx.doi.org/10.1016/j.fss.2016.05.010

Please cite this article in press as: L. De Miguel et al., Pointwise aggregation of maps: Its structural functional equation and some applications to social choice theory, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.05.010

Having these ideas in mind, we wonder what information we need in order to obtain the aggregating map f_{n+1}. Given an element $x \in X$, to compute $f_{n+1}(x)$ sometimes we need to know the maps $f_1, \dots, f_n$ at every point of the domain, or at least at several points different from x. But it may also happen that in order to get $f_{n+1}(x)$ we only need to have at hand the values of $f_1, \dots, f_n$ at that point x. That is: $f_{n+1}(x)$ directly comes from $f_1(x), \dots, f_n(x)$, and we may then assume that there exists a map $G : Y^n \to Y$ such that $f_{n+1}(x) = G(f_1(x), \dots, f_n(x))$ holds for every $x \in X$. Then we say that f_{n+1} depends *pointwise* on the collection $\{f_1, f_2, \dots, f_n\}$.

Let us consider the following illustrative example:

Example 1. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ denote two real valued functions of one single real variable. Let $f_3, f_4 : \mathbb{R} \to \mathbb{R}$ respectively be defined, for every $x \in \mathbb{R}$, through the functional equations $f_3(x) = f_1(x) + f_2(x)$ and $f_4(x) = f_1(2x) + f_2(3x)$. Despite the fact that both functional equations look similar at first glance, from a structural point of view they are quite different. The reason is that, working with the former one, in order to know the value of the map f_3 at a point $x \in \mathbb{R}$ we only need to know the values that f_1 and f_2 take *at the same point* x. However, in the second equation, to determine $f_4(x)$ it is not enough to know the values of $f_1(x)$ and $f_2(x)$. As a matter of fact, we need to know the values of f_1 and f_2 at *point(s) different from* x.

This nuance is essential in our approach throughout this paper. The first case corresponds to the so-called *pointwise aggregation* of maps. The second case, despite still being an aggregation of maps, cannot be called "pointwise", at least a priori.

In the present paper, given two abstract (nonempty) sets X, Y as well as a natural number n and n maps $\{f_1, f_2, \dots, f_n : X \to Y\}$, we study how to aggregate those maps to obtain a new one, say, $f_{n+1} : X \to Y$, in a way such that the value of f_{n+1} at a point $x \in X$ depends only on the values $\{f_1(x), f_2(x), \dots, f_n(x)\}$ at the same point x. This is what we call *a pointwise aggregation* of $\{f_1, \dots, f_n\}$.

The structure of the paper is as follows:

In Section 2 we formalize the notion of pointwise aggregation of maps. In Section 3 we study the main structural functional equation linked to the pointwise aggregations of maps. The particular case of real-valued functions is studied in Section 4. In Section 5 we analyze a relevant case study, namely type-2 fuzzy sets, via fusion operators. In Section 6 we discuss several applications to social choice theory. We conclude with Section 7, which includes suggestions for further research.

2. Basic concepts and notation

Definition 1. Let X, Y denote two (nonempty) sets. Let n be a natural number. Let Y^X denote the set of maps from X into Y, that is, $Y^X = \{f : X \to Y\}$. Let $(f_1, \dots, f_n) \in (Y^X)^n$ stand for an n-tuple of maps from X into Y. A map $f_{n+1} \in Y^X$ is said to be:

(i) An *aggregation* of $(f_1, \dots, f_n)$ if there exists a map $T : (Y^X)^n \to Y^X$ such that $f_{n+1} = T(f_1, \dots, f_n)$. In this case the map T is said to be an *n-dimensional aggregation operator*.[1]

(ii) A *pointwise aggregation* of $(f_1, \dots, f_n)$ if there exists a map $W : Y^n \to Y$ such that $f_{n+1}(x) = W(f_1(x), \dots, f_n(x))$ holds for every $x \in X$. In this case, the map W is said to be a *pointwise n-dimensional aggregator*, whereas the functional equation $f_{n+1}(x) = W(f_1(x), \dots, f_n(x))$ is said to be the *structural functional equation of pointwise aggregation of maps*.

[1] Notice that our definition of "n-dimensional aggregation operator" is less restrictive than the usual ones that are often encountered in the literature of fuzzy sets and related topics (see e.g. [15,9]). We have decided to keep it for the sake of completeness, bearing in mind that possible applications in other settings–perhaps unrelated to fuzzy set theory–could result from this more general definition. (See e.g. the applications in mathematical social choice theory analyzed in Section 6 of this manuscript.)

Please cite this article in press as: L. De Miguel et al., Pointwise aggregation of maps: Its structural functional equation and some applications to social choice theory, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.05.010