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Entropy for non-additive measures in continuous domains

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Abstract

In a recent paper we introduced a definition of f-divergence for non-additive measures. In this paper we use this result to give a definition of entropy for non-additive measures in a continuous setting. It is based on the KL divergence for this type of measures. We prove some properties and show that we can use it to find a measure satisfying the principle of minimum discrimination.

Keywords: Entropy; KL-divergence; Non-additive measures

1. Introduction

Entropy is an important concept in information theory defined for probability distributions. It is used to measure the difference of the quantity of information before and after a data transmission. It is also used in statistics to help to define a probability distribution under some constraints. This is done through the application of the maximum entropy principle [2]. For example, the Gaussian distribution maximizes the entropy over all distributions with the same variance (see e.g. [2] p. 255 and Ch. 12). For continuous probability distributions the principle of minimum discrimination is used, which is based on the Kullback–Leibler divergence [11].

Non-additive measures [21,3,19] (also known as capacities and as fuzzy measures) generalize additive measures, and thus probabilities, replacing additivity by monotonicity. They have been applied in a large variety of contexts (computer vision, decision making, economics).

At present there exist several generalizations [24,12] of the entropy for discrete non-additive measures. See also [8] for an overview. Nevertheless, up to our knowledge there are no definitions available for measures in continuous domains. In this paper we focus in this problem.

We introduce a definition for non-additive measures for infinite sets. The measure roots in our recent definition [23] of the f-divergence for non-additive measures.

The structure of the paper is as follows. In Section 2, we present some definitions needed later on and in Section 3 we introduce our definition and give some properties. Section 4 generalizes the principle of minimum discrimination to non-additive measures. The paper finishes with some conclusions and lines of future work.

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2. Preliminaries

In this section we review some results on non-additive measures and integrals, divergences on measures, and entropies. See [19,21,3,5,18] for details.

2.1. Non-additive measures and the Choquet integral

A non-additive measure is a monotonic set function.

Definition 1. Let $(\Omega, \mathcal{F})$ be a measurable space. A set function μ defined on $\mathcal{F}$ is called a non-additive measure (or fuzzy measure) if an only if

1. $0 \leq \mu(A) \leq \infty$ for any $A \in \mathcal{F}$;
2. $\mu(\emptyset) = 0$;
3. If $A_1 \subseteq A_2 \subseteq \mathcal{F}$ then

$$\mu(A_1) \leq \mu(A_2)$$

Distorted Lebesgue measures, introduced in [7], are an example of non-additive measures. They are defined in terms of the Lebesgue measure λ and a distortion function. The distortion function should be non-decreasing. Then, μ is a distorted Lebesgue measure if it can be expressed as $\mu = m \circ \lambda$ where m is a non-decreasing distortion function and λ is the Lebesgue measure. Recall that the Lebesgue measure of an interval $[a, b]$ is $\lambda([a, b]) = b - a$.

Some other types of measures are useful in this paper.

Definition 2. Given a non-additive measure μ,

1. we say that μ is submodular if

$$\mu(A) + \mu(B) \geq \mu(A \cup B) + \mu(A \cap B),$$

2. we say that μ is supermodular if

$$\mu(A) + \mu(B) \leq \mu(A \cup B) + \mu(A \cap B).$$

The Choquet integral of a function with respect to a non-additive measure is defined below. Using the Choquet integral, we can consider the derivative of a non-additive measure with respect to another.

Definition 3. [1] Let $(\Omega, \mathcal{F})$ be a measurable space and let $\nu, \mu : \mathcal{F} \to \mathbb{R}^+$ be non-additive measures. We say that ν is a Choquet integral of μ if there exists a measurable function $g : \Omega \to \mathbb{R}^+$ with

$$\nu(A) = (C) \int_A g d\mu \tag{1}$$

for all $A \in \mathcal{F}$.

In this paper we need the following proposition related to the Choquet integral.

Theorem 1. *[1,3,5] Let μ be a non-additive measure on $(\mathbb{R}, \mathcal{B})$, and f, g be non negative measurable functions. Then, the following properties hold.*

1. *When μ is submodular, then*

$$(C) \int (f + g) d\mu \leq (C) \int f d\mu + (C) \int g d\mu.$$