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Block-wise construction of commutative increasing monoids

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Abstract

Construction of new associative, commutative and increasing operations on the unit interval from a given operation $*$ is proposed. It is shown that when $*$ is a t-norm, t-conorm or a proper uninorm, new t-norms, t-conorms and proper uninorms can be obtained.

Keywords: Associative operation; t-Conorm; t-Norm; Uninorm

1. Introduction

Associative operations are a useful tool in fuzzy set theory, but also in many areas of application, especially in decision-making under uncertainty (see, e.g., [3]), image processing (see, e.g., [4]), in fuzzy neural networks (see, e.g., [5]), etc. The most important classes of commutative increasing monoids in the framework of fuzzy sets are those of t-norms and t-conorms. In this area we would like to point out at least two monographs – by Klement, Mesiar and Pap [8] and by Schweizer and Sklar [10]. Later Jenei [6] introduced a new construction method for t-norms. As a technical tool he introduced the notion of a t-subnorm.

Another important class of commutative increasing monoids is that of uninorms. Uninorms appeared for the first time by Dombi [3] under the name 'aggregative operators'. Dombi's aggregative operators were constructed with the aim to fuzzify evaluation of objects in the theory of multicriteria decision-making. With the help of aggregative operators, objects can be divided into two classes – those which satisfy given criteria at least at a threshold level α, and those which do not. Aggregative operators introduced by Dombi are nowadays known under the name representable uninorms.

Independently of Dombi, also in the paper by Czogała and Drewniak [2] uninorms (but also other associative operations) were studied. Later uninorms were re-introduced by Yager and Rybalov [11] as a generalization of both t-norms and t-conorms. There are several families of uninorms. For an overview of basic families see, e.g., [9].

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In [7] the authors introduced a construction method that enables to define new monotone associative and commutative operations $\oplus\colon [0,1]^2 \to [0,1]$ from a given monotone associative and commutative operation $*\colon [0,1]^2 \to [0,1]$. This method is called paving. The main idea of paving is that the unit interval is split into countably many disjoint sub-intervals $(I_i)_{i\in J}$, where J is an index-set. Further a family of increasing transformations $\varphi_i\colon I_i \to [0,1]$ is given. The new operation $\oplus$ (constructed via paving) is then defined by

$$x \oplus y = \varphi^{-1}_{i\otimes j}(\varphi_i(x) * \varphi_j(y)), \quad \text{where } x \in I_i \text{ and } y \in I_j \tag{1}$$

and $\otimes$ is an appropriate operation on the index-set J ($\otimes$ might be, e.g., addition). Details about paving will be explained in the next section. Using this construction method, a uninorm that is strictly increasing on $]0,1[^2$ (but not continuous) was constructed in [7].

In this paper we will modify the above mentioned method in such a way that we will consider splitting the unit interval into only finitely many sub-intervals. Further, we will be looking for properties of possible operations on J that could be used instead of addition.

2. Preliminaries

In this section we recall some known notions and facts to make the paper self-contained.

Let X be a non-empty set. An algebraic structure $(X,\odot)$ is a *monoid* if $\odot$ is an associative binary operation on X with a neutral element.

A function $N\colon [0,1] \to [0,1]$ is said to be a *negation* if N is decreasing and $N(0)=1$, $N(1)=0$.

A negation N is said to be *strong* if it is *involutive*, i.e., if $N(N(x))=x$ for all $x\in[0,1]$.

Definition 2.1. (see, e.g., [8,10]) A *triangular norm* $T\colon [0,1]^2 \to [0,1]$ (t-norm for short) is a commutative, associative and increasing operation, whose neutral element is 1.

Remark 2.2. Note that, for a strong negation N, the N-dual operation to a t-norm T defined by $S(x,y) = N(T(N(x),N(y)))$ is called a t-conorm. For more information, see, e.g., [8].

Definition 2.3 *([6])*. A *triangular subnorm* $T_S\colon [0,1]^2 \to [0,1]$ (t-subnorm for short) is a commutative, associative, increasing binary operation fulfilling the condition $T_S(x,y) \le \min\{x,y\}$ for all $(x,y)\in[0,1]^2$.

Dually, *triangular superconorm* $S_S\colon [0,1]^2 \to [0,1]$ (t-superconorm for short) is a commutative, associative, increasing binary operation fulfilling the condition $T_S(x,y) \ge \max\{x,y\}$ for all $(x,y)\in[0,1]^2$.

Details and more facts on t-norms and t-conorms can be found in [8,10].

Definition 2.4 *([11])*. A *uninorm* $U\colon [0,1]^2 \to [0,1]$ is a commutative, associative and increasing operation with arbitrary neutral element e.

If $e \in]0,1[$, we say that the uninorm U is proper.

If $U(1,0)=0$ holds, U is called conjunctive. If $U(1,0)=1$ holds, U is called disjunctive. Conjunctive and disjunctive uninorms are dual to each other. For an arbitrary disjunctive uninorm U and a strong negation N its N-dual conjunctive uninorm is given by

$$U^d_N(x,y) = N(U(N(x),N(y)))\,.$$

For an overview of basic properties on uninorms we refer to [1].

Definition 2.5. Let $*\colon [0,1]^2 \to [0,1]$ be a commutative operation. Fix a value $a\in[0,1]$. We say that $x\in[0,1]$, $x\neq a$, is an *a-divisor* if there exists $y\in[0,1]$, $y\neq a$ such that

$$x * y = a.$$

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