ARTICLE IN PRESS
FSS:7127

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

On the existence of aggregation functions with given super-additive and sub-additive transformations

Alexandra Šipošová *, Ladislav Šipeky, Jozef Širáň

Faculty of Civil Engineering, Slovak University of Technology, 810 05 Bratislava, Slovakia

Received 3 January 2016; received in revised form 31 October 2016; accepted 23 November 2016

Abstract

In this note we study restrictions on the recently introduced super-additive and sub-additive transformations, $A \mapsto A^*$ and $A \mapsto A_*$, of an aggregation function A. We prove that if A^* has a slightly stronger property of being strictly directionally convex, then $A = A^*$ and A_* is linear; dually, if A_* is strictly directionally concave, then $A = A_*$ and A^* is linear. This implies, for example, the existence of pairs of functions $f \le g$ sub-additive and super-additive on $[0, \infty[^n$, respectively, with zero value at the origin and satisfying relatively mild extra conditions, for which there exists no aggregation function A on $[0, \infty[^n$ such that $A_* = f$ and $A^* = g$.

Keywords: Aggregation function; Sub-additive and super-additive transformation

1. Introduction

Aggregation functions turn out to play an important role in applications to situations where information expressed in the form of a collection of values needs to be merged into a single value. Examples of such instances include decision making based on aggregating scores or preferences relative to a given set of alternatives and compressing information by fusing inputs from several sources to facilitate recognition and classification, with applications in risk management, artificial intelligence, and many other areas of human activities and technology.

Literature on aggregation functions is abundant and we just refer to [2,5] for basic facts. Needless to say that various types of aggregation functions have been introduced and examined (cf. [2,5] again). We will consider aggregation functions on the domain $[0, \infty[^n$, although a number of other domains have been considered, depending on applications. For the purpose of this article, an *aggregation function* is a mapping $A: [0, \infty[^n \to [0, \infty[$ which is increasing in every coordinate and such that $A(\mathbf{0}) = A(0, \dots, 0) = 0$.

We will focus on a pair of mutually dual fundamental transformations of aggregation functions, which have been introduced in [6] and, again, motivated by interesting applications in economics.

* Corresponding author.
E-mail addresses: siposova@math.sk (A. Šipošová), sipeky@math.sk (L. Šipeky), siran@math.sk (J. Širáň).

http://dx.doi.org/10.1016/j.fss.2016.11.009
0165-0114/© 2016 Published by Elsevier B.V.

Definition 1. Let $A : [0,\infty[^n \to [0,\infty[$ be an aggregation function. The sub-additive transformation $A_* : [0,\infty[^n \to [0,\infty]$ of A is given by

$$A_*(\mathbf{x}) = \inf \{\sum_{i=1}^{k} A(\mathbf{x}^{(i)}) \ \mid \ \sum_{i=1}^{k} \mathbf{x}^{(i)} \geq \mathbf{x}\} \tag{1}$$

where $\mathbf{x}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ are vectors from $[0,\infty[^n$.

Similarly, the super-additive transformation $A^* : [0,\infty[^n \to [0,\infty]$ of A is defined by

$$A^*(\mathbf{x}) = \sup \{\sum_{j=1}^{k} A(\mathbf{x}^{(j)}) \ \mid \ \sum_{j=1}^{k} \mathbf{x}^{(j)} \leq \mathbf{x}\} \ . \tag{2}$$

We note that the transformations (1) and (2) were originally introduced in [6] for aggregation functions as defined above *and* with the property that A_* and A^* do not attain the value ∞. It is easy to show [6] that the functions A_* and A^* are indeed, as their names suggest, sub-additive and super-additive, respectively, that is, $A_*(\mathbf{u}+\mathbf{v}) \leq A_*(\mathbf{u}) + A_*(\mathbf{v})$ and $A^*(\mathbf{u}+\mathbf{v}) \geq A^*(\mathbf{u}) + A^*(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v} \in [0,\infty[^n$, where addition is defined coordinate-wise in the usual manner.

Determination of which functions $A : [0,\infty[^n \to [0,\infty[$ satisfy $A = A^*$ or $A = A_*$ is important in application in economics. As explained in [6], the values of an aggregation function A at $\mathbf{x}$ may represent the output generated subject to a vector $\mathbf{x}$ of production factors, and then, for every vector $\overline{\mathbf{x}}$ of available resources the theoretically optimal output is given exactly by the value of $A^*(\overline{\mathbf{x}})$. Similarly, if $A(\mathbf{x})$ represents a price for a set of n-tuples of goods with quantities given by the coordinates of $\mathbf{x}$, then a theoretically optimal purchase price of a preassigned set of goods given by a vector $\overline{\mathbf{x}}$ is given by the value of $A_*(\overline{\mathbf{x}})$. Impacts of considering the cases when $A = A^*$ and $A = A_*$ for such type of applications in economics are also discussed in [6].

Further examples of relevance of the study of various aspects of aggregation function can be found in [7] by introducing the concepts of $(A, \mathcal{D})$-based sub-decomposition and super-decomposition integrals that depend on a given aggregation function A and on a decomposition system of its domain. It turns out that, for instance, the value of $A^*(\mathbf{x})$ is equal to the value of the $(A, \mathcal{D})$-based sub-decomposition integral at $\mathbf{x}$ if $\mathcal{D}$ is a decomposition system of the domain of A consisting of singletons. The same paper also discusses a number of applications of sub- and super-decomposition integrals in economics (regarding, for example, work distribution planning and optimizing costs in making orders from a preassigned set of products), which makes determination of A^* and A_* for a given aggregation function A relevant also from this point of view.

It is obvious that $A_* = A^*$ if and only if A is additive (i.e., A is a weighted sum of coordinates of the input vector), and then $A_* = A^* = A$. In the opposite case, we have $A_* < A^*$, and identification of any further relations between A_* and A^* is completely open. This suggests the question of whether or not for every pair $f, g : [0,\infty[^n \to [0,\infty]$ such that $f(\mathbf{0}) = g(\mathbf{0}) = 0$, $f(\mathbf{x}) \leq g(\mathbf{x})$ for every $\mathbf{x} \in [0,\infty[^n$, with f sub-additive and g super-additive, there exists an aggregation function A on $[0,\infty[^n$ such that $A_* = f$ and $A^* = g$.

In this paper we show that the answer to this question is negative if relatively mild extra conditions are imposed on f and g. This is a consequence of our findings stemming from a more detailed study of replacing the super- and sub-additivity properties of the above transformations by the slightly stronger properties of directional convexity and concavity. Our main results say in a nutshell that if an aggregation function A is such that A^* is strictly directionally convex, then necessarily $A = A^*$ and A_* is linear; dually, if A_* is strictly directionally concave, then $A = A_*$ and A^* is linear.

We explain our tools in section 2 on the one-dimensional case first, as it clearly distinguishes what hurdles one needs to overcome in extending the result to the multi-dimensional case, which is done in section 3. We conclude in section 4 by a discussion.

2. The one-dimensional case

We consider here the one-dimensional case, a preliminary report on which can be found in [12]. To explain why this case is considered separately, we begin by observing that the notion of an aggregation function was extended in [5] to provide a framework for functions with neutral element and for means, with boundary conditions and monotonicity as