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Short Communication

T-subnorms with strong associated negation: Some properties

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Received 21 September 2016; received in revised form 7 December 2016; accepted 22 January 2017

Abstract

In this work we investigate t-subnorms M that have strong associated negation. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, nilpotent elements, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations lead us to two open problems which are also presented.

Keywords: T-norms; T-subnorms; Archimedeanness; Conditional cancellativity; Left-continuity; R-implications; Residual implications

1. Introduction

The theory of triangular norms and triangular subnorms has been well studied and its applications well-established. Many algebraic and analytical properties of these operations, viz., Archimedeanness, conditional cancellativity, left-continuity, etc., have been studied and their inter-relationships shown (see for instance, [6]).

Yet another way of categorizing t-subnorms is as follows: Given a t-subnorm M, one can obtain its associated negation n_M (see Definitions 2.2 and 2.4 below). Note that n_M is usually not a fuzzy negation, i.e., $n_M(1) \geq 0$. However, we can broadly consider two sub-classes of t-subnorms based on whether their associated negation n_M is strong or not.

In this work, we study the class of t-subnorms whose associated negation n_M is strong. Firstly, we show that such t-subnorms are necessarily t-norms. Following this, we investigate some particular classes of these and study the inter-relationships between different algebraic and analytic properties of such t-subnorms, viz., Archimedeanness, conditional cancellativity, left-continuity, etc. In particular, we show that under this setting many of these properties are equivalent. Our investigations have led us to two open problems, which are also presented.

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http://dx.doi.org/10.1016/j.fss.2017.01.012

2. Preliminaries

To make this short note self-contained, we present some important definitions and properties, which can be found in [6,1].

Definition 2.1. A fuzzy negation is a function $N: [0,1] \to [0,1]$ that is non-increasing and such that $N(1) = 0$ and $N(0) = 1$. Further, it is said to be strong or involutive, if $N \circ N = id_{[0,1]}$.

Definition 2.2. A t-subnorm is a function $M: [0,1]^2 \to [0,1]$ such that it is monotonic non-decreasing, associative, commutative and $M(x,y) \leq \min(x,y)$ for all $x, y \in [0,1]$, i.e., 1 need not be the neutral element.

Definition 2.3. Let M be a t-subnorm.

(i) If 1 is the neutral element of M, then it becomes a t-norm. We denote a t-norm by T in the sequel.
(ii) M is said to satisfy the Conditional Cancellation Law if, for any $x, y, z \in (0,1]$,

$$M(x,y) = M(x,z) > 0 \text{ implies } y = z\,. \tag{CCL}$$

Alternately, (CCL) implies that on the positive domain of M, i.e., on the set $\{(x,y) \in (0,1]^2 \mid M(x,y) > 0\}$, M is strictly increasing.
(iii) M is said to be *Archimedean*, if for all $x, y \in (0,1)$ there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} < y$.
(iv) An element $x \in (0,1)$ is a *nilpotent* element of M if there exists an $n \in \mathbb{N}$ such that $x_M^{[n]} = 0$.
(v) A t-norm T is said to be *nilpotent*, if it is continuous and if each $x \in (0,1)$ is a nilpotent element of T.

Definition 2.4. Let M be any t-subnorm and $x, y \in [0,1]$.

- The R-implication I_M of M is given by

$$I_M(x,y) = \sup\{t \in [0,1] \mid M(x,t) \leq y\}\,. \tag{1}$$

- The associated negation n_M of M is given by

$$n_M(x) = \sup\{t \in [0,1] \mid M(x,t) = 0\}. \tag{2}$$

A brief note on nomenclature is perhaps warranted here. Firstly, the R-implication I_M will be termed a *residual* implication only if the underlying t-subnorm M is left-continuous.

Secondly, while n_M is clearly a non-increasing function and $n_M(0) = 1$, note that it need not be a fuzzy negation, since $n_M(1)$ can be greater than 0. Hence, only in the case n_M is a fuzzy negation we call n_M the *natural negation* of M in this work. However, many results hold even if $n_M(1) > 0$, see for instance [3,9], and hence to preserve this generality in such situations we term n_M as the *associated negation*.

For instance, the following result is true even when $n_M(1) > 0$.

Proposition 2.5 *(cf. [1], Proposition 2.3.4). Let M be any t-subnorm and n_M its associated negation. Then we have the following:*

(i) $M(x,y) = 0 \Longrightarrow y \leq n_M(x)$.
(ii) $y < n_M(x) \Longrightarrow M(x,y) = 0$.
(iii) *If M is left-continuous then $y = n_M(x) \Longrightarrow M(x,y) = 0$, i.e., the reverse implication of (i) also holds.*

Proposition 2.6. *Let M be any t-subnorm with n_M being a natural negation with e as its fixed point, i.e., $n_M(e) = e$. Then*

(i) *Every $x \in (0,e)$ is a nilpotent element; in fact, $x_M^{[2]} = 0$ for all $x \in [0,e)$.*
(ii) *In addition, if M is either conditionally cancellative or left-continuous, then e is also a nilpotent element.*

Please cite this article in press as: B. Jayaram, T-subnorms with strong associated negation: Some properties, Fuzzy Sets Syst. (2017), http://dx.doi.org/10.1016/j.fss.2017.01.012