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Shock models with dependence and asymmetric linkages

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Abstract

This paper introduces a new class of copulas and shows its relevance for applications. In particular, a stochastic interpretation in terms of a system of dependence components affected by a global shock is given. As a main feature of the model, the global shock has an opposite effect on the different components of the system. Copulas generated by this mechanism are characterized in the bivariate case and their main properties are illustrated. Connections with concepts like semilinear copulas and conic aggregation functions are also highlighted. Moreover, a high dimensional extension is presented.

Keywords: Copula; Dependence concepts; Marshall–Olkin distribution; Shock models; Tail dependence

1. Introduction

Dependence concepts play a crucial role in multivariate statistical literature since it was recognized that the independence assumption cannot describe conveniently the behavior of a stochastic system. Since then, different attempts have been made in order to provide more flexible methods to describe the variety of dependence-types that may occur in practice. Copula models have become popular in different applications in view of their ability to describe the relationships among random variables in a flexible way. To this end, several families of copulas have been introduced, motivated by special needs from the scientific practice (see, for instance, [17,20,33]).

Consider, for instance, the case when one wants to build a stochastic model for describing the dependence among two (or more) lifetimes, i.e. positive random variables. In engineering applications, joint models of lifetimes may serve to estimate the expected lifetime of a system composed by several components. In a related situation like portfolio credit risk, instead, the lifetimes may have the interpretation of time-to-default of firms, or generally financial entities, while a stochastic model may estimate the price/risk of a relate derivative contract (e.g. CDO). In both cases, it is of

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interest to estimate the probability of the occurrence of a joint default, which means, in the case of a bivariate random vector (X, Y), the probability of the event $\{X = Y\}$, or more generally $\{f(X) = g(Y)\}$ for some measurable functions f and g. Obviously, if one requires the event $\{f(X) = g(Y)\}$ to have non-zero probability, then the copula for (X, Y) must have a singular component, as described in [10,28].

The generation of convenient statistical distribution for modeling such situations originated from the seminal paper by Marshall–Olkin [30]; see [2] for an up-to-date overview. In [12], a general framework was introduced for such constructions, which is briefly recalled here.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space. For $d \geqslant 2$, consider a system composed by d components whose behavior is described by the continuous random variables (=r.v.'s) $X_1, \dots, X_d$ such that each X_i is distributed according to a continuous distribution function F_i, $X_i \sim F_i$. The r.v. X_i can be interpreted as a shock that effects only the i-th component of the system, i.e. the idiosyncratic shock. Let $\mathcal{S} \neq \emptyset$ be a collection of subsets $S \subseteq \{1, 2, \dots, d\}$ with at least 2 elements. For each $S \in \mathcal{S}$ consider the random variable Z_S with probability distribution function G_S. These r.v.'s Z_S can be interpreted as an (exogenous) shock that may affect the stochastic behavior of all the system components with index $i \in S$, i.e. the systemic shock. Furthermore, we assume a given dependence among the introduced random vectors $\mathbf{X}$ and $\mathbf{Z}$. To this end, according to Sklar's theorem [35], we assume that there exists a copula C such that

$$(\mathbf{X}, \mathbf{Z}) \sim C((F_i)_{i=1,\dots,d}, (G_S)_{S\in\mathcal{S}}).$$

The copula C hence describes how the shocks $\mathbf{X}$ and $\mathbf{Z}$ are related. Finally, for $i = 1, \dots, d$, assume the existence of a *linking function* Ψ_i that expresses how the effects produced by the shock X_i and all the shocks Z_S with $i \in S$ are combined together and act on the i-th component. Given the previous framework, the d-dimensional stochastic model $\mathbf{Y} = (Y_1, \dots, Y_d)$ can be constructed by setting, for $i = 1, \dots, d$,

$$Y_i = \psi_i(X_i, Z_{S:i\in S}).$$

Interestingly, under suitable assumptions on the d.f.'s F_i's and G_S's, the d.f. of $\mathbf{Y}$ is given by a copula (for more details, see [11]). Several families of copulas can be interpreted by using the previous stochastic mechanism.

- If $(\mathbf{X}, \mathbf{Z})$ is a vector of independent components, then the resulting copula is of Marshall–Olkin type, according to different generalizations provided by [29–31] and by [4,6,15,16] for the exchangeable case.
- If $\mathbf{X}$ and $\mathbf{Z}$ are independent vectors, but there is some dependence among the components of $\mathbf{X}$, then the resulting copula has been described in [11,12] in the case only one exogenous shock affects the system.
- Furthermore, if a specific dependence is assumed between $\mathbf{X}$ and $\mathbf{Z}$, then several constructions have been provided in [32] (see also [8]) and [3].

Actually, in all previously cited examples, the linking function ψ_i does not change with i, i.e. the exogenous shocks affect all the components of the system in the same way. One attempt to weaken this assumption has been provided in [34]. Here, the main idea is to consider three independent r.v.'s X_1, X_2 and Z that are used to construct the bivariate vector (Y_1, Y_2) such that $Y_1 = \max\{X_1, Z\}$ and $Y_2 = \min\{X_2, Z\}$. The resulting copula has been characterized in [34, Theorem 9], and is also called *maxmin copula*, since it considers maximum and minimum as linking functions.

Various interpretations of this model can be found, since it is possible in many practical situations that the common exogenous shock will produce different effects on different system components. For instance, we may think of X_1 and X_2 as r.v.'s representing the respective wealth of two groups of people, and the exogenous shock Z is interpreted as an event that is beneficial to the first group and detrimental to the second one. Analogously, X_1 and X_2 can be thoughts as a short and a long investment, respectively, while Z is beneficial only to one of this type of investment.

One of the main goals of this paper is to extend the latter model in order to allow the two underlying system variables X_1 and X_2 to be dependent. In particular, the dependence is assumed to be governed by a general copula, while the third variable Z is assumed to be independent of both X_1 and X_2. Specifically, we prove that joint d.f. arising from the previously described stochastic mechanism is actually a copula (section 2) and we illustrate some of its main features (section 3). Finally, we discuss possible multivariate generalizations (section 4).

As a matter of fact, the obtained class of copulas may have some properties that are appealing in various contexts related to fuzzy set theory and multicriteria decision making. First, it includes non-exchangeable copulas, which are used for instance as more general fuzzy connectives. See, for instance, [1,14]. Then, its associated measure may have

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