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Fuzzy relation equations and inequalities with two unknowns and their applications ☆

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Abstract

In this paper we study several types of systems of fuzzy relation equations and inequalities composed of a given family of fuzzy relations between two sets and two unknown fuzzy relations on these sets. Solutions to these systems are pairs of fuzzy relations on the underlying sets and can be ordered coordinatewise. We show that solutions to each of these systems form a complete lattice, and provide procedures for computing the greatest solution which is less than or equal to a given pair of fuzzy relations. We also demonstrate the application of solutions to these systems in data reduction.

Keywords: Fuzzy relation; Residuals of fuzzy relations; Fuzzy relation equation; Two-mode fuzzy social network; Fuzzy formal context; Data reduction

1. Introduction

Methods for solving various systems of fuzzy relation equations and inequalities, as well as their application in solving practical problems, were the subject of numerous research articles. The study of such systems was initiated by E. Sanchez, who used them in medical research (cf. [37–39]). Later they found a much wider field of application, and nowadays they are used in fuzzy control, discrete dynamic systems, knowledge engineering, identification of fuzzy systems, prediction of fuzzy systems, decision-making, fuzzy information retrieval, fuzzy pattern recognition, image compression and reconstruction, fuzzy automata theory, fuzzy social network analysis and in many other areas (cf., e.g., [7,11,12,16,17,23,28,30,31,41]).

Sanchez started the study of the so-called *linear systems* that consist of equations and inequalities with one side containing the composition of an unknown fuzzy relation and a given fuzzy relation or fuzzy set, while the other side contains only another given fuzzy relation or fuzzy set. Solvability and methods for computing the greatest

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solutions to linear systems of fuzzy relation equations and inequalities over various structures of truth values have been investigated in numerous papers, including those over complete residuated lattices (cf., e.g., [5,6,11,27,32,33,35,36]). More complex systems that consist of equations and inequalities whose both sides contain compositions of an unknown fuzzy relation and given fuzzy relations, called *weakly linear*, have been recently studied in [20–22], and equations and inequalities with one side containing the composition of an unknown fuzzy set and a given fuzzy relation, while the other side contains only the unknown fuzzy set, have been studied in [24–26,29,34,40]. In a more general context similar equations have been considered in [18].

In this paper we study a different kind of systems that consist of equations and inequalities whose both sides contain compositions of one given fuzzy relation between two sets and two different unknown fuzzy relations on these sets. Solutions of such a system are those pairs of fuzzy relations that fulfill all equations and inequalities in this system and they can be ordered coordinatewise. We consider various types of such systems, including the ones that also contain inverses of the unknown fuzzy relations, for any of these systems we prove that its solutions form a complete lattice, and we provide a procedure for computing the greatest solution which is less than or equal to a given pair of fuzzy relations. The greatest solutions to the considered systems of fuzzy relation inequalities can be expressed in a simple way, without the need for using of any iterative procedure. In contrast, the greatest solutions to the considered systems of fuzzy relation equations are computed using an iterative method which is closely related to the so-called *alternating method* developed by Cuninghame-Green and Zimmermann [9] in the context of solving equations on partially ordered sets defined by residuated functions (see also [8]). What is common to these two methods is the construction of a descending sequence of pairs of elements or fuzzy relations, which terminates under certain conditions and determines the greatest solution we were looking for. However, the termination criteria proposed in [9] are specific for partially ordered sets and are not applicable to fuzzy relations, and here we provide termination criteria that are specific for fuzzy relations. Moreover, we perform an analysis of the computational time of the proposed algorithm. We also provide an algorithm for computing the greatest crisp solutions to the considered systems that can be used in cases where the above mentioned procedure does not terminate in a finite number of steps and can not be used to efficiently compute the greatest fuzzy solutions to these systems. The algorithm is based on the concept of Boolean residuals of fuzzy relations.

We also discuss the possible fields of application of the considered systems of fuzzy relation equations and inequalities. First of all, we have in mind two potential interpretations of the considered fuzzy relational systems and the corresponding systems of fuzzy relation equations and inequalities: the underlying sets can be understood as a set of actors and a set of related events, as when dealing with two mode networks in the social network analysis, or as a set of objects and a set of their attributes, as in the concept data analysis and relational database research. For any bipartite fuzzy relational system we define the concept of a quotient fuzzy relational system with respect to a pair of fuzzy quasi-orders on the underlying sets, and we characterize those quotient fuzzy relational systems that are determined by pairs of fuzzy quasi-orders and fuzzy equivalences which are solutions to the considered systems of fuzzy relation equations and inequalities. These characterizations explain how the pairs of fuzzy quasi-orders and fuzzy equivalences which are solutions to the considered systems can be used for simultaneous reduction of actors and events in two-mode fuzzy networks, simultaneous reduction of objects and attributes in fuzzy formal contexts, and for other related purposes.

The paper is organized as follows. In Section 2 we recall basic notions and notation concerning fuzzy sets, fuzzy relations and residuals of fuzzy relations. The main results of the paper are presented in Section 3, and in Section 4 we discuss the applications of these results and give several computational examples.

2. Preliminaries

2.1. Fuzzy sets

In this paper we use complete residuated lattices as structures of membership values. A *residuated lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\to$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \to z. \tag{1}$$