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On the construction of adjunctions between a fuzzy preposet and an unstructured set

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Abstract

In this work, we focus on adjunctions, also called isotone Galois connections, in the framework of fuzzy preordered sets (hereafter, fuzzy preposets). Specifically, we present necessary and sufficient conditions so that, given a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preposet $\mathbb{A}$ into an unstructured set B, it is possible to construct a suitable fuzzy preorder relation on B for which there exists a mapping $g\colon B \to \mathbb{A}$ such that the pair (f, g) constitutes an adjunction.

Keywords: Galois connection; Adjunction; Preorder; Fuzzy sets

1. Introduction

The notion of adjunction (or its sibling Galois connection) can be encountered in several research areas, both from a practical and a theoretical point of view. In the literature, one can find numerous papers on theoretical developments on adjunctions [1,2,6,7,12,17] and also on applications thereof [9,10,19–22,24].

Bělohlávek [1] introduced a fuzzy generalization of the notion of Galois connection and, since then, several papers have appeared on further approaches to either fuzzy adjunctions or fuzzy Galois connections; see [3,10–12,17,18,26] for some recent contributions. In some cases, a specific approach is introduced with a particular purpose in mind: for instance, Shi et al. [23] focused on the notion of fuzzy adjunction in view of its use in fuzzy mathematical morphology.

In [25,26], fuzzy Galois connections on fuzzy posets were introduced as a generalization of Bělohlávek's fuzzy Galois connection, and our approach in this paper is precisely based on this generalization. Specifically, we are in-

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terested in constructing a right adjoint (or residual mapping) associated to a given mapping $f: \langle A, \rho_A \rangle \to B$ from a fuzzy preposet $\langle A, \rho_A \rangle$ into an unstructured set B. Of course, a convenient fuzzy preorder relation has to be defined on B.

In previous works [14,15], some of the present authors have studied this problem in the crisp case for a mapping $f: \langle A, \leq_A \rangle \to B$ from a partially (pre)ordered set A, and also in the fuzzy case, where the approach was extended to a fuzzy poset $\langle A, \rho_A \rangle$. However, it has been argued that the antisymmetry property of fuzzy order relations is rather restrictive [4,5], and should be weakened to a version involving a given fuzzy equivalence relation. From that point of view, fuzzy preorder relations are the most natural candidates, as they come along with their own fuzzy equivalence relation (the symmetric kernel relation). A brief survey on the definition of fuzzy posets can be found in [27].

The aim of the paper is to consider a mapping $f: \mathbb{A} \to B$ from a fuzzy preposet $\mathbb{A} = \langle A, \rho_A \rangle$ into an unstructured set B, and then characterize those situations in which B can be endowed with a fuzzy preorder relation and an isotone mapping $g: B \to A$ can be built such that the pair (f, g) becomes an adjunction. This problem is more than a mere exercise in generalization since antisymmetry, in practice, is usually a too strong requirement.

Although all the results will be stated in terms of the existence and construction of right adjoints (or residual mappings), all of them can be straightforwardly modified for the existence and construction of left adjoints (or residuated mappings). On the other hand, it is worth to remark that the construction developed in this paper can be extended to the different types of adjunctions (or Galois connections) between fuzzy preposets (see [13]).

The structure of the paper is as follows. In Section 2, the preliminary notions used in the rest of the paper are introduced. Then, the characterization of the existence of right adjoint, together with its construction is given in Section 3. Finally, in Section 4, we state the conclusions and prospects for future work.

2. Preliminary definitions

The most common underlying structure for considering fuzzy generalizations of Galois connections is that of a complete residuated lattice $\mathbb{L} = (L, \leq, \top, \bot, \otimes, \to)$. We will denote the supremum and infimum operation in the lattice with the symbols $\vee$ and $\wedge$, respectively.

An $\mathbb{L}$-fuzzy set on U is a mapping $X: U \to L$ where $X(u)$ denotes the degree to which u belongs to X; the *core* of X is the (crisp) set of elements $a \in A$ such that $X(a) = \top$.

Let X and Y be $\mathbb{L}$-fuzzy sets, X is said to *be included in* Y, denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$. The union (resp. intersection) of X and Y is defined as the $\mathbb{L}$-fuzzy set $(X \cup Y)(u) = X(u) \vee Y(u)$ (resp. $(X \cap Y)(u) = X(u) \wedge Y(u)$) for each $u \in U$.

A binary $\mathbb{L}$-fuzzy relation R on U is an $\mathbb{L}$-fuzzy subset of $U \times U$, i.e. $R: U \times U \to L$, and it is said to be:

(i) *Reflexive* if $R(a, a) = \top$, for all $a \in U$.
(ii) $\otimes$-*Transitive* if $R(a, b) \otimes R(b, c) \leq R(a, c)$, for all $a, b, c \in U$.
(iii) *Symmetric* if $R(a, b) = R(b, a)$, for all $a, b \in U$.
(iv) *Antisymmetric* if $R(a, b) = R(b, a) = \top$ implies $a = b$, for all $a, b \in U$.

The corresponding generalizations of preorder, order, and equivalence relation are the usual ones, namely[3]:

(i) An $\mathbb{L}$-*fuzzy preorder relation* is a fuzzy relation that is reflexive and $\otimes$-transitive.
(ii) An $\mathbb{L}$-*fuzzy order relation* is a fuzzy relation that is reflexive, antisymmetric and $\otimes$-transitive.
(iii) An $\mathbb{L}$-*fuzzy equivalence relation* is a fuzzy relation that is reflexive, symmetric and $\otimes$-transitive.

Definition 1.

(i) A *fuzzy preposet* (fuzzy preposet) is a pair $\mathbb{U} = \langle U, \rho_U \rangle$ in which U is a set and ρ_U is a fuzzy preorder relation on U.

[3] From now on, when no confusion arises, we will omit the prefixes $\mathbb{L}$ and $\otimes$.

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