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Fixed points of adjoint functors enriched in a quantaloid

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Received 2 January 2016; received in revised form 30 November 2016; accepted 2 December 2016

Abstract

Representation theorems are established for fixed points of adjoint functors between categories enriched in a small quantaloid. In a very general setting these results set up a common framework for representation theorems of concept lattices in formal concept analysis (FCA) and rough set theory (RST), which not only extend the realm of formal contexts to multi-typed and multi-valued ones, but also provide a general approach to construct various kinds of representation theorems. Besides incorporating several well-known representation theorems in FCA and RST as well as formulating new ones, it is shown that concept lattices in RST can always be represented as those in FCA through relative pseudo-complements of the given contexts, especially if the contexts are valued in a non-Girard quantaloid.

Keywords: Quantaloid; Adjoint $\mathcal{Q}$-functors; $\mathcal{Q}$-distributor; Isbell adjunction; Kan adjunction; Concept lattice; Formal concept analysis; Rough set theory

1. Introduction

This paper aims to establish general representation theorems for fixed points of adjoint functors between categories enriched in a small quantaloid $\mathcal{Q}$, which set up a common framework for representation theorems of concept lattices in formal concept analysis (FCA) [4,6] and rough set theory (RST) [17,18] in the generality of their $\mathcal{Q}$-version. As Galois connections between posets are precisely adjoint functors between categories enriched in the two-element Boolean algebra **2**, we start the introduction from this classical case.

A Galois connection [4] $s \dashv t$ between posets C, D consists of monotone maps $s : C \longrightarrow D$, $t : D \longrightarrow C$ such that $s(x) \leq y \iff x \leq t(y)$ for all $x \in C$, $y \in D$. By a fixed point of $s \dashv t$ is meant an element $x \in C$ with $x = ts(x)$ or, equivalently, an element $y \in D$ with $y = st(y)$, since

$$\mathsf{Fix}(ts) := \{x \in C \mid x = ts(x)\} \quad \text{and} \quad \mathsf{Fix}(st) := \{y \in D \mid y = st(y)\}$$

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http://dx.doi.org/10.1016/j.fss.2016.12.001
0165-0114/© 2016 Published by Elsevier B.V.

Please cite this article in press as: H. Lai, L. Shen, Fixed points of adjoint functors enriched in a quantaloid, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.12.001

are isomorphic posets with the inherited order from C and D, respectively. As the first main result of this paper, the following theorem characterizes those posets representing $\mathsf{Fix}(ts) \cong \mathsf{Fix}(st)$:

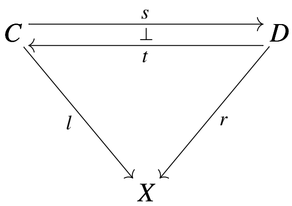
Theorem 1.1. *Let $s \dashv t : D \longrightarrow C$ be a Galois connection between posets. A poset X is isomorphic to $\mathsf{Fix}(ts)$ if, and only if, there exist surjective maps $l : C \longrightarrow X$ and $r : D \longrightarrow X$ such that*

$$\forall c \in C, \forall d \in D : \ s(c) \leq d \text{ in } D \iff l(c) \leq r(d) \text{ in } X.$$

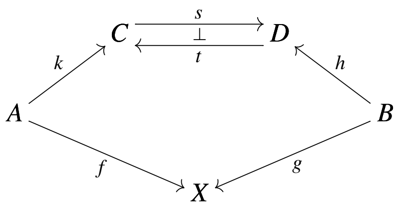
It is well known that if C, D are complete lattices, then so is $\mathsf{Fix}(ts) \cong \mathsf{Fix}(st)$. In this case, the above representation theorem can be strengthened to the following one, which is our second main result, in terms of $\bigvee$-dense and $\bigwedge$-dense maps:

Theorem 1.2. *Let $s \dashv t : D \longrightarrow C$ be a Galois connection between complete lattices. A complete lattice X is isomorphic to $\mathsf{Fix}(ts)$ if, and only if, there exist $\bigvee$-dense maps $f : A \longrightarrow X$, $k : A \longrightarrow C$ and $\bigwedge$-dense maps $g : B \longrightarrow X$, $h : B \longrightarrow D$ such that*

$$\forall a \in A, \forall b \in B : \ sk(a) \leq h(b) \text{ in } D \iff f(a) \leq g(b) \text{ in } X.$$



Theorem 1.1



Theorem 1.2

These two theorems play the role of general representation theorems and their power will be revealed when being applied to concept lattices. To see this, recall that given a relation $\varphi : A \nrightarrow B$ between sets (usually called a *formal context*, or *context* for short, and written as (A, B, φ) in FCA and RST), there are two Galois connections

$$\varphi_{\uparrow} \dashv \varphi^{\downarrow} : (\mathbf{2}^B)^{\mathrm{op}} \longrightarrow \mathbf{2}^A \quad \text{and} \quad \varphi^* \dashv \varphi_* : \mathbf{2}^A \longrightarrow \mathbf{2}^B \tag{1.i}$$

given by

$$\varphi_{\uparrow}(U) = \{y \in B \mid \forall x \in U : \ x\varphi y\}, \quad \varphi^{\downarrow}(V) = \{x \in A \mid \forall y \in V : \ x\varphi y\},$$
$$\varphi^*(V) = \{x \in A \mid \exists y \in V : \ x\varphi y\}, \quad \varphi_*(U) = \{y \in B \mid \forall x \in A : \ x\varphi y \Longrightarrow x \in U\}$$

for all $U \subseteq A$, $V \subseteq B$; the complete lattices consisting of their fixed points,

$$\mathsf{M}\varphi := \mathsf{Fix}(\varphi^{\downarrow}\varphi_{\uparrow}) \quad \text{and} \quad \mathsf{K}\varphi := \mathsf{Fix}(\varphi_*\varphi^*),$$

are respectively (up to isomorphism) the *concept lattices* of the context (A, B, φ) in FCA and RST.[2] The *fundamental theorem* of FCA characterizes those complete lattices representing $\mathsf{M}\varphi$:

Theorem 1.3. *(See [4,6].) A complete lattice X is isomorphic to $\mathsf{M}\varphi$ if, and only if, there exist a $\bigvee$-dense map $f : A \longrightarrow X$ and a $\bigwedge$-dense map $g : B \longrightarrow X$ such that*

$$\forall a \in A, \ \forall b \in B : \ a\varphi b \iff f(a) \leq g(b) \text{ in } X.$$

The following diagrams explain how one derives the above theorem from 1.1 and 1.2:

[2] $\mathsf{M}\varphi$ and $\mathsf{K}\varphi$ are also called the *formal concept lattice* and the *object-oriented concept lattice* of the context (A, B, φ), respectively.

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