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# A non-frame valued Cartesian closed category of liminf complete fuzzy orders

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**Abstract**

Let $H = \{0, \frac{1}{2}, 1\}$ with the natural order and $p\&q = \max\{p + q - 1, 0\}$ for all $p, q \in H$. It is proved that the category of liminf complete $H$-ordered sets is Cartesian closed. It reveals that there exists a Cartesian closed category consisting of non-frame valued liminf complete fuzzy ordered sets.


*Keywords:* Fuzzy relations; $L$-order; Liminf complete $L$-ordered set; Cartesian closed category

## 1. Introduction

In 1971, Zadeh introduced the notion of fuzzy preorders [23]. Later, this notion was extended to the setting that the truth-value table is a complete residuated lattice $L$ and the concept of $L$-ordered sets was introduced in [3]. Categorically speaking, an $L$-ordered set is just an $L$-enriched category [12,13,18,19]. Since this notion was introduced to the study of domain theory [7,10,13], many basic concepts such as direct completeness, continuity and Scott topology in domain theory have been extended to the $L$-valued setting [13,18–22,24]. Thus, $L$-ordered sets give an important approach to the study of Quantitative Domain theory [7,17,18].

Searching for Cartesian closed categories is one of the basic problems in domain theory. Turn to the $L$-valued setting, based on the condition that $L$ being a frame, several Cartesian closed categories have been obtained [14,15,22]. In [14], Lai and Zhang proved that when the truth-value lattice $L$ is a frame the category **Liminf**($L$) of liminf complete $L$-ordered sets is Cartesian closed, however, if $L$ is a complete residuated lattice then this category is seldom Cartesian closed. In fact, it is proved that if $L = ([0, 1], \&)$ with $\&$ being a continuous t-norm, then the category **Liminf**($L$) is Cartesian closed if and only if $\& = \wedge$ [14]. So, a natural question is whether there exists a complete residuated lattice $L$ such that $L$ is not a frame but the category of liminf complete $L$-ordered sets is Cartesian closed.

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In [14], it is shown that if $H$ is the three-valued MV-algebra, then the category of $H$-ordered sets is Cartesian closed. In this note, we show that the category of liminf complete $H$-ordered sets is also Cartesian closed. Thus, there does exist a complete residuated lattice $L$ that is not a frame but the category of liminf complete $L$-ordered sets is Cartesian closed. The problem to characterize those complete residuated lattices $L$ for which **Liminf**$(L)$ is Cartesian closed still remains open.

## 2. Cartesian closedness of Liminf($H$)

Let $L$ be a complete residuated lattice [3]. Since this paper is a continuation of [14], in order to make the paper more concise, we keep all notions about liminf complete $L$-ordered sets to agree with [14]. For concepts and results in domain theory and category theory, we refer to [1,2,9].

For convenience, we recall that a net $\alpha = \{x_\lambda\}_{\lambda\in D}$ in an $L$-ordered set $A$ is *forward Cauchy*, if $1 = \bigvee_{\lambda\in D}\bigwedge_{\lambda\le\mu\le\sigma} A(x_\mu, x_\sigma)$. A forward Cauchy net $\{x_\lambda\}_{\lambda\in D}$ in an $L$-ordered set $A$ converges to $a \in A$, or $a$ is a liminf of $\{x_\lambda\}_{\lambda\in D}$, if for all $x\in A$, $A(a,x) = \bigvee_{\lambda\in D}\bigwedge_{\lambda\le\mu} A(x_\mu, x)$. We denote the liminf of $\alpha = \{x_\lambda\}_{\lambda\in D}$ by $\liminf \alpha$ or $\liminf_{\lambda\in D} x_\lambda$ if it exists. An $L$-ordered set $A$ is *liminf complete* if each forward Cauchy net has a liminf.

The complete residuated lattice considered in this note is the MV-algebra $H$ with three elements. Precisely, $H = \{0, \frac{1}{2}, 1\}$ and $p\&q = \max\{p+q-1, 0\}$ for all $p, q \in H$. We note that the Łukasiewicz conjunction operation $p\&q = \max\{p+q-1, 0\}$ is the same as the nilpotent minimum $p * q = \min(p, q)$ if $p > 1-q$ and 0 otherwise. This well-known operation was introduced in [8] and also appears in [16]. For other conjunctions on 3-valued structures, we refer to [4].

It is trivial that $(H, \wedge)$ is a frame [11]. The right adjoint for $p\&\_$ and $p\wedge\_$ will be denoted by $p \overset{\&}{\rightarrow} \_$ and $p \overset{\wedge}{\rightarrow} \_$, respectively.

Some properties of $H$-ordered sets are collected below for later use.

- Let $A$ be an $H$-ordered set and $\{x_\lambda\}_{\lambda\in D}$ a monotone net in the underlying ordered set $A_0$. Then $\bigvee_{\lambda\in D}\bigwedge_{\lambda\le\mu} A(x_\mu, x) = \bigwedge_{i\in D} A(x_i, x)$ for all $x\in A$. This shows that if $\liminf_{\lambda\in D} x_\lambda$ exists, then $\liminf_{\lambda\in D} x_\lambda$ equals the supremum of the net $\{x_\lambda\}_{\lambda\in D}$ in $A_0$. In particular, the underlying ordered set $A_0$ of a liminf complete $H$-ordered set $A$ is directed complete.
- In an $H$-ordered set $A$, a net $\{x_\lambda\}_{\lambda\in D}$ is forward Cauchy if and only if it is eventually monotone in $A_0$ because 1 is a compact element in $H$.
- Let $\{x_\lambda\}_{\lambda\in D}$ be a forward Cauchy in an $H$-ordered set $A$. Take some $\lambda_0 \in D$ such that $x_u \ge x_\lambda$ whenever $\mu \ge \lambda \ge \lambda_0$. Let $D' = \{\lambda\in D | \lambda \ge \lambda_0\}$. Then $\{x_\lambda\}_{\lambda\in D}$ converges if and only if $\{x_\lambda\}_{\lambda\in D'}$ converges, in this case $\liminf_{\lambda\in D} x_\lambda = \liminf_{\lambda\in D'} x_\lambda$.

**Lemma 2.1.** *Let $A$, $B$ be two $H$-ordered sets, $f : A\to B$ an order preserving map. Then*

(1) *$A$ is liminf complete if and only if every monotone net in $A_0$ converges in $A$.*
(2) *If $A$ is liminf complete then $f$ is liminf continuous if and only if $f$ preserves liminfs of monotone nets if and only if $f : A_0 \to B_0$ is Scott continuous.*

Let $[A, B]$ be the set of order preserving maps from $A$ to $B$. The function space $[A, B]$ is defined by $[A, B](f, g) = \bigwedge_{x\in A} B(f(x), g(x))$. For the sake of unambiguity, we denote this $L$-order by sub in this paper. Denote the set of all liminf continuous functions from $A$ to $B$ by $[A \to B]$. Then $([A \to B], \mathrm{sub})$ is a sub-$L$-ordered set of $([A, B], \mathrm{sub})$. If $B$ is a liminf complete $L$-ordered set, then $([A \to B], \mathrm{sub})$ is liminf complete and the liminf of a forward Cauchy net $\{f_\lambda\}_{\lambda\in D}$ in $([A \to B], \mathrm{sub})$ is given by $f(x) = \liminf_{\lambda\in D} f_\lambda(x)$ [13,14]. Thus, $([A \to B], \le_{\mathrm{sub}})$ is closed in the set of Scott-continuous functions from $A_0$ to $B_0$ under the formation of directed sups.

Let $A$, $B$ be two $H$-ordered sets. Define $P : [A, B] \times [A, B] \longrightarrow H$ as follows:

$$P(f, g) = \bigvee\{p \in H | \forall x, y \in A, p \wedge A(x, y) \le B(f(x), g(y))\}.$$

Then $P$ is an $H$-order on $[A, B]$ (see [6]). Denote the $H$-ordered set $([A, B], P)$ by $[A, B]_P$. Define $ev : [A, B]_P \times A \longrightarrow B$ by $ev(f, x) = f(x)$. Then $ev$ is order preserving (see [5,6]). Restricting to the set of liminf continuous