ARTICLE IN PRESS
JID:FSS AID:7121 /FLA [m3SC+; v1.238; Prn:25/11/2016; 8:50] P.1 (1-14)

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

Finite fuzzy topological spaces

Moussa Benoumhani, Ali Jaballah *

Department of Mathematics, The University of Sharjah, Sharjah, P.O. Box: 27272, United Arab Emirates

Received 6 February 2016; received in revised form 12 November 2016; accepted 15 November 2016

Abstract

We compute for the first time, the number of fuzzy topologies defined on a finite set and having a small number of open sets. Certain cases, where the number of open sets is large, are also considered. Several well known results are obtained as corollaries. The paper is ended by some questions for future investigations.

Keywords: Fuzzy topology; Number of fuzzy topologies; Combinatorics; Topology

1. Introduction

A fuzzy set as defined by Lotfi Zadeh is a function from a set X on $[0, 1]$. This definition has been extended to more general sets than the unit interval; for example to a complete lattice. Let X be a set, M be a totally ordered one, and let $\mathcal{F} = \mathcal{F}(X, M) = M^X$ be the collection of fuzzy subsets of X with membership values in M, or equivalently, the set $\mathcal{F}$ of functions from X to M.

$\mathcal{F}$ is partially ordered by:

$$\mu \preceq \nu \Longleftrightarrow \mu(x) \leq \nu(x) \text{ for every } x \in X.$$

This set is also a complete lattice with the same partial order. We also have

$$\mu \prec \nu, \text{ if and only if } \mu \preceq \nu \text{ and } \mu(x) < \nu(x), \text{ for some } x \in X.$$

The fuzzy subsets $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$ are the functions defined respectively by $0_{\mathcal{F}}(x) = 0$ for every $x \in X$, and $1_{\mathcal{F}}(x) = 1$ for every $x \in X$. For every fuzzy subset μ different from $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$, we have $0_{\mathcal{F}} \prec \mu \prec 1_{\mathcal{F}}$.

A fuzzy topology τ on the set X, as defined in [8], is a collection of fuzzy subsets of X such that $0_{\mathcal{F}}$ and $1_{\mathcal{F}}$ are in τ, $\sup_{i\in I} u_i$ is in τ for every $(u_i)_{i\in I} \in \tau$, and $\min\{u_i, u_j\}$ is in τ for every u_i and u_j in τ. Note that a fuzzy topology on $\mathcal{F}$ is just a sublattice of $\mathcal{F}$, containing the least and the greatest elements. Since its definition, fuzzy topology has

* Corresponding author. Fax: +971 65050318.
E-mail address: ajaballah@sharjah.ac.ae (A. Jaballah).

http://dx.doi.org/10.1016/j.fss.2016.11.003

Please cite this article in press as: M. Benoumhani, A. Jaballah, Finite fuzzy topological spaces, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.11.003

attracted many researchers. See [10,24,26], and [33] for recent investigations in this field. On the other hand fuzzy topological spaces on finite sets have not been considered yet.

The number of finite topologies in the classical case is an outstanding and open problem. There is no known explicit formula for the total number of topologies $T(n)$, one can define on an n-element set. Yet, there are some results. In fact the bijective correspondence between topologies and quasiorders (relations which are reflexive and transitive) on a finite set is known since the thirties of the last century; it was discovered independently by Alexandroff [1] and Birkhoff [6]. The antisymmetric quasiorders are called orders, they correspond to T_0 topologies; recall that a topology is T_0, if for every $x \neq y$, there is an open set containing exactly one of them. It is also known that there is a correspondence between transitive graphs (or 0–1 matrices) and finite topologies, [13] and [12]. This result was used by Evans et al. [13] to compute $T(n)$ for $2 \leq n \leq 7$. The sequence $T(n)$ is known just for $n \leq 18$, see [29]. Using the matrix representation of a finite topology, Krishnamurty [22] obtained the upper bound $T(n) \leq 2^{n(n-1)}$. Asymptotically and up to homeomorphism, the total number of topologies on a set of cardinality n is $2^{\frac{n^2}{4}}$. This result is due to D. Kleitman and B. Rothschild [19].

Another approach of the subject is the enumeration according to the number of open sets. Let $T(n,k)$ be the number of topologies on an n-element set having k $(2 \leq k \leq 2^n)$ open sets. It is known that $T(n,k) = 0$, for $3 \cdot 2^{n-2} < k < 2^n$ and $n \geq 3$. Stanley [30] computed $T(n,k)$ for $2 \leq k \leq 5$ and for $k \geq 7 \cdot 2^{n-4}$. In computing these values, Stanley used quasiorders (and orders instead of finite topologies), and hence the open sets are just the ideals of the order (or quasiorder). This approach was fruitful for some small values as well as for some large values of k, and the limits of this method were reached. Then, the only way to extended $T(n,k)$ for $k \leq 17$ and $k \geq 5 \cdot 2^{n-4}$ was to move from the other equivalent concepts of a finite topology (quasiorder, matrix or a graph) and use the original definition of a topology; its very rich structure supplies tools that are not existing in the other notions and are easier to handle. This enabled the authors of [4,20,21] to push the calculations of $T(n,k)$ for other values of k.

On the other hand, fuzzy topological spaces satisfying some finiteness conditions have not been considered yet unlike the classical topology where this field is still active and attracting several researchers by its importance and by the numerous long-standing unsolved problems, [3,4,7]. Also, finite topology has several applications such as image processing, which is concerned with visual information; it creates, stores, manipulates and displays digital images. In all these operations finite topology is involved. For example, we would like to know to what degree an image (a photo, for example) is an exact reflexion of the real world image. Also, after a digital transformation of a picture, we need the topological aspects of the image intact.

To answer these questions (and others), the plane $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ is endowed with a topology on it called the digital topology. The points (k,l), where k and l are odd numbers, are the pixels and form the minimal open sets of this topology. The subspace of the digital plane consisting of all open points is called the visible screen. This corresponds to the pixels in a digital image display. Any finite number of points of the visible screen is a finite subspace. The investigations of these finite topologies supply information about the image display and the other questions related to image processing.

Another application is in biology; where the mutation of RNA molecules is investigated. The model is a finite topological space based on a probability distribution of a mutation. For more details about the previous and other examples see [1].

Finite topology is also present in chemistry. An approach to molecular topology is based on the atomic adjacency, so any topology modeling a hydrocarbon molecule must contain a bond as an open set. In other words if atoms a and b are connected, then the set {a, b} is open. This topology is the smallest one generated by the subbasis of all bonds of 2-element set of atoms. This is the bond topology. Any space in the bond topology is a product of its components, so it has the form $R^r S^s S'^{s'} S''^{s''}$ where r, s, s', s'' are the number of the components of each type. Once we have the form of the topology, we can restitute the general chemical formula of the molecule. For other topologies defined on the molecule see [23].

The enumeration of finite fuzzy structures, and the combinatorial aspect in general in the fuzzy realm is curiously rare, except some sporadic papers dealing with fuzzy subgroups of finite fuzzy groups and fuzzy ideals of finite rings such as [14,32] and the references therein.

The main purpose of this work is to remedy this lack and initiate the corresponding fuzzy side of these problems. In [27] the digital topology, known in image processing is shown to be deficient, and another (somewhat mixed) model is presented.

Please cite this article in press as: M. Benoumhani, A. Jaballah, Finite fuzzy topological spaces, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.11.003