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Ideals and involutive filters in generalizations of fuzzy structures

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Abstract

(Bounded integral) residuated lattices (which need not be commutative) form a large class of algebras containing some classes of algebras behind many-valued and fuzzy logics. Congruences of such algebras are usually defined and investigated by means of their normal filters. In the paper we introduce and investigate ideals of residuated lattices. We show that one can define, in some cases, congruences also using ideals and that the corresponding quotient residuated lattices are involutive.

Keywords: Residuated lattice; Involutive residuated lattice; Pseudo *BL*-algebra; Filter; Normal filter; Ideal

1. Introduction

Bounded integral residuated lattices (= residuated lattices) form a large class of algebras containing some classes of algebras behind many-valued and fuzzy logics, such as: *MV*-algebras [2], i.e. an algebraic counterpart of the Łukasiewicz infinite valued logic, *BL*-algebras [14], which are algebras of Hájek's basic (fuzzy) logic, *MTL*-algebras [10], i.e. algebras of the monoidal t-norm based logic. A special class of residuated lattices properly containing the class of *BL*-algebras is formed by so-called commutative $R\ell$-monoids [8]. Moreover, Heyting algebras [1] which are algebras of the intuitionistic logic can be also considered as residuated lattices.

The class of residuated lattices also contains the classes of non-commutative variants of the above kinds of algebras: *GMV*-algebras [17] (or equivalently pseudo *MV*-algebras [13]), pseudo *BL*-algebras [5,6], pseudo *MTL*-algebras [11] and $R\ell$-monoids [9].

In the paper, we will investigate algebraic structure properties of residuated lattices.

All mentioned algebras, except for *GMV*-algebras and *MV*-algebras, use multiplication as the initial binary operation and so the sets of filters (or deductive systems) in connection with congruences of such algebras are intensively investigated. In contrast, *GMV*-algebras and *MV*-algebras use addition as the initial binary operation, and thus the investigation of structure properties of *GMV*-algebras and *MV*-algebras is oriented on the sets of their ideals. Note

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that ideals of many algebraic structures (e.g. lattices, rings, lattice ordered groups) play a fundamental role in the corresponding theories.

In the theory of *GMV*- and *MV*-algebras it is possible to define multiplication as the dual operation to addition. Consequently, one can introduce also filters there, which are duals of ideals, and hence the filter and ideal theories of *GMV*- and *MV*-algebras are mutually dual.

But in general, a dual binary operation to multiplication in residuated lattices does not exist. Consequently, a notion of "the (precise) dual to filter" does not exist too. Nevertheless, in [16] a kind of an ideal of a *BL*-algebra (which need not be an *MV*-algebra) has been introduced and it was shown that such ideals are very useful in the study of structure properties of *BL*-algebras. Among others, it is possible to define quotient *BL*-algebras not only using filters but in particular cases also using ideals. Namely, quotient *BL*-algebras induced by ideals are in fact *MV*-algebras.

In this paper we introduce and investigate ideals of general residuated lattices (which need not be commutative). For this, we define two kinds of binary operations (left and right addition) and use them to introducing left, right and both-sided ideals (= ideals) of residuated lattices. We show that every ideal of a residuated lattice M induces an equivalence which is a congruence on a reduct of M, and it is a congruence on M in the case of a pseudo *BL*-algebra. Moreover, the corresponding quotient residuated lattice is involutive and in the case of pseudo *BL*-algebras it is a *GMV*-algebra. Note that if a residuated lattice M is an *MV*-algebra (resp. a *GMV*-algebra) then the ideals of the residuated lattice $M = (M;\ \odot, \vee, \wedge, \rightarrow, 0, 1)$ (resp. $M = (M;\ \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$) coincide with ideals (resp. normal ideals) of the *MV*-algebra (resp. *GMV*-algebra) $M = (M;\ \oplus, ^{-}, 0, 1)$ (resp. $M = (M;\ \oplus, ^{-}, ^{\sim}, 0, 1)$).

Further we show that for any good residuated lattice M satisfying the Glivenko property (GP), involutive filters are precisely normal filters that contain the filter $D(M)$ of dense elements in M. Using this result, we describe a one-to-one correspondence between ideals and involutive (normal) filters of any pseudo *BL*-algebra.

2. Preliminaries

A *bounded integral residuated lattice* is an algebra $M = (M;\ \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$ satisfying the following conditions:

(i) $(M;\ \odot, 1)$ is a monoid;
(ii) $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice;
(iii) $x \odot y \le z$ iff $x \le y \rightarrow z$ iff $y \le x \rightsquigarrow z$ for any $x,\ y \in M$.

In what follows, by a *residuated lattice* we will mean a bounded integral residuated lattice. If the operation $\odot$ on a residuated lattice M is commutative then M is called *commutative residuated lattice*. In such a case the operations $\rightarrow$ and $\rightsquigarrow$ coincide. In a residuated lattice M we define two unary operations (negations) $^{-}$ and $^{\sim}$ as follows: $x^{-} = x \rightarrow 0$, $x^{\sim} = x \rightsquigarrow 0$ for each $x \in M$.

Recall that the mentioned algebras of many-valued and fuzzy logics are characterized in the class of residuated lattices as follows. A residuated lattice M is

– a pseudo *MTL*-algebra if M satisfies the identities of pre-linearity
 (iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;
– an $R\ell$-monoid if M satisfies the identities of divisibility
 (v) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;
– a pseudo *BL*-algebra if M satisfies both (iv) and (v);
– involutive if M satisfies the identities
 (vi) $x^{-\sim} = x = x^{\sim-}$;
– a *GMV*-algebra (or equivalently a pseudo *MV*-algebra) if M satisfies (iv), (v) and (vi);
– a Heyting algebra if the operations $\odot$ and $\wedge$ coincide.

A residuated lattice M is called *good*, if it satisfies the identity $x^{-\sim} = x^{\sim-}$. For example, every commutative residuated lattice, every *GMV*-algebra and every pseudo *BL*-algebra which is a subdirect product of linearly ordered pseudo *BL*-algebras [7] is good.

In the next propositions we recall some basic properties of residuated lattices.