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The lattice of prefilters of an EQ-algebra

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Abstract

In this paper, the notion of a prefilter generated by a nonempty subset of an EQ-algebra is introduced and a characterization of it is obtained. It is proved that the set of all prefilters of an EQ-algebra is an algebraic lattice and it is a Brouwerian lattice for an ℓEQ-algebra. Furthermore, it is shown that the set of all principal prefilters of an ℓEQ-algebra is a sublattice of the lattice of prefilters. Then by defining an implication between two prefilters, it is determined that the lattice of prefilters is a Heyting algebra, for an ℓEQ-algebra. Finally, the EQ-algebras for which the lattice of prefilters is a Boolean algebra are given.

Keywords: EQ-algebra; Boolean algebra; Heyting algebra

1. Introduction

Fuzzy logic deals with non-Boolean truth-values, not uncertainty per se [16]. Residuated lattice, BL-logic [9], MTL-logic [7], R_0-logic [14,16,10,18] and Godel logic [8], etc. are well-known fuzzy logics. The operations in these algebras are $\otimes$, $\to$, $\wedge$ and $\vee$, in which fuzzy equality $\leftrightarrow$ is derived from $\otimes$, $\wedge$ and $\to$ [2,7,9,10,15,14,17].

V. Novák and B. De Baets introduced a special algebra called EQ-algebra in [3–5,13]. These algebras are intended to develop an algebraic structure of truth values for fuzzy type theory.

EQ-algebras are interesting and important algebras for studying and researching. Residuated lattices and BL-algebras are particular cases of EQ-algebras.

An EQ-algebra has three binary operations (meet, multiplication and a fuzzy equality) and a top element and also a binary operation (implication) is derived from fuzzy equality. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. From the potential application point of view, it seems very interesting that unlike [9], we can have non-commutativity without having to introduce two kinds of implication.

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The prefilter theory plays a fundamental role in the general development of EQ-algebras. Implicative and positive implicative prefilters in EQ-algebras were defined by Lianzhen Liu and Xiangyang Zhang [11].

In this paper, we characterize a prefilter of an EQ-algebra generated by a nonempty set and find a lot of prefilters by this. We hope that these prefilters open a new door into the theory of prefilters in EQ-algebras. Then for an EQ-algebra ξ, we denote by $PF(E)$ the set of all prefilters of ξ and present some known definitions and results relative to $PF(E)$. In particular, we find conditions under which the lattice $PF(E)$ is a Heyting and Boolean algebra.

This paper is organized as follows: in section 2, we review the basic definitions, properties and special types of EQ-algebras. In section 3, we define and characterize prefilters generated by nonempty subsets of an EQ-algebra. We also determine prefilter generated by meet and join of two elements and show that the set of all principal prefilters of an ℓEQ-algebra is a sublattice of the algebraic lattice of all prefilters of an EQ-algebra. Finally, by defining $\rightarrow$ between two prefilters, we determine that $(PF(E), \vee, \wedge, \rightarrow, <1>)$ is a Heyting algebra, for an ℓEQ-algebra.

2. Preliminaries

Definition 2.1. [6] An algebra $\xi = (E, \wedge, \otimes, \sim, 1)$ of type $(2, 2, 2, 0)$ is called an EQ-algebra where for all $a, b, c, d \in E$:

$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $a \le b$ iff $a \wedge b = a$,
$(E2)$ $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \le b$,
$(E3)$ $a \sim a = 1$, (reflexivity axiom)
$(E4)$ $(a \wedge b) \sim c) \otimes (d \sim a) \le c \sim (d \wedge b)$, (substitution axiom)
$(E5)$ $(a \sim b) \otimes (c \sim d) \le (a \sim c) \sim (b \sim d)$, (congruence axiom)
$(E6)$ $(a \wedge b \wedge c) \sim a \le (a \wedge b) \sim a$, (monotonicity axiom)
$(E7)$ $a \otimes b \le a \sim b$,
for all $a, b, c \in E$.

The binary operations $\wedge$, $\otimes$ and $\sim$ are called meet, multiplication and a fuzzy equality, respectively.

Clear, $(E, \le)$ is a partial order. We will also put, for $a, b \in E$

$$\widetilde{a} = a \sim 1 \text{ and } a \rightarrow b = (a \wedge b) \sim a$$

The binary operation $\rightarrow$ will be called implication.

If 0 is a bottom element of E, then we may define the unary operation $\neg$ on E, for all $a \in E$, by

$$\neg a = a \sim 0.$$

If E is a nonempty set with three binary operations $\wedge, \otimes, \sim$ such that $(E, \wedge, 1)$ is a $\wedge$-semilattice, $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone with respect to $\le$, then $(E, \otimes, \wedge, \sim, 1)$ is an EQ-algebra, where $a \sim b = 1$, for all $a, b \in E$.

Lemma 2.2. [6] *Let ξ be an* EQ-*algebra. Then the following properties hold for all $a, b, c, d \in E$:*

(e_1) $a \sim b = b \sim a$,
(e_2) $(a \sim b) \otimes (b \sim c) \le (a \sim c)$,
(e_3) $(a \rightarrow b) \otimes (b \rightarrow c) \le (a \rightarrow c)$ *and* $(b \rightarrow c) \otimes (a \rightarrow b) \le (a \rightarrow c)$,
(e_4) $a \sim d \le (a \wedge b) \sim (d \wedge b)$,
(e_5) $(a \sim d) \otimes ((a \wedge b) \sim c) \le (d \wedge b) \sim c$,
(e_6) $(a \wedge b) \sim a \le (a \wedge b \wedge c) \sim (a \wedge c)$,
(e_7) $a \otimes b \le a \wedge b \le a, b$,
(e_8) $b \le \widetilde{b} \le a \rightarrow b$,
(e_9) $(a \rightarrow b) \otimes (b \rightarrow a) \le a \sim b \le (a \rightarrow b) \wedge (b \rightarrow a)$,
(e_{10}) *If* $a \le b$, *then* $a \rightarrow b = 1$, $b \rightarrow a = a \sim b$, $\widetilde{a} \le \widetilde{b}$, $c \rightarrow a \le c \rightarrow b$ *and* $b \rightarrow c \le a \rightarrow c$,
(e_{11}) *If* $a \le b \le c$, *then* $a \sim c \le a \sim b$ *and* $a \sim c \le b \sim c$,
(e_{12}) $a \otimes (a \sim b) \le \widetilde{b}$,
(e_{13}) $[c \rightarrow (a \wedge b)] \otimes (a \sim d) \le c \rightarrow (d \wedge b)$,
(e_{14}) *If* $a \le b \rightarrow c$ $(a \le b \sim c)$, *then* $a \otimes b \le \widetilde{c}$,
(e_{15}) $(a \rightarrow b) \otimes (c \rightarrow d) \le (a \wedge c) \rightarrow (b \wedge d)$,