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Minimal solutions of general fuzzy relation equations on linear carriers. An algebraic characterization ☆

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Abstract

This paper considers a general fuzzy relation equation, which has minimal solutions, if it is solvable. In this case, an algebraic characterization is introduced which provides an interesting method to compute minimal solutions in this general setting. Moreover, a comparison with other frameworks is also given.

Keywords: Fuzzy relation equations; Minimal solutions; Residual structures

1. Introduction

Supremum-⊙ fuzzy relation equations were introduced by E. Sanchez [22] in the seventies, in order to investigate theoretical and applicational aspects of fuzzy set theory [11]. Several generalizations of the original equations have been introduced, such as [3,6,12]. These and many other papers study the existence of solutions of these equations [1,5,10], and, in the affirmative case, they show that the set of solutions is a upper-preserving complete lattice in which the greatest solutions can easily be obtained. However, to know about minimal solutions is more difficult. Indeed, in particular cases, these solutions do not exist.

The existence of the greatest solution is fundamental in order to obtain an operative mechanism to compute the whole set of solutions or, at least, that this set will easily be determined. It is interesting to fix a general framework in which the minimal solutions of each solvable fuzzy relation equation exist and that each solution will be between the greatest solution and a minimal solution.

Several papers [4,8,17,23,30,20,21,31,25] have studied how these solutions can be obtained, in restrictive frameworks and several algorithms have been developed. However, these restrictions limit the flexibility of the possible

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applications. This paper considers a general setting, in which the operators may neither be commutative nor associative and they only need to be monotone and residuated inf-preserving mappings of non-empty sets on the right argument. This last property is not restrictive. The linearity of the carrier, together with the inf-preserving property, ensures the existence of minimal solutions whenever a solution exists.

Moreover, an algebraic characterization of these solutions is given, which provides a mechanism to obtain the minimal solutions. Before that, two illustrative examples will be introduced for a better understanding of the method.

Furthermore, a comparison with other frameworks in past publications is given, which shows that the considered setting is more general in order to ensure minimal solutions and to provide a method to compute them. Hence, this can be used in a larger range of applications.

The structure of the paper is as follows: Section 2 introduces a general setting, from which a method to obtain the minimal solutions of a solvable fuzzy relation equation is then given in Section 3. Section 5 presents a comparison with other frameworks and the paper finishes with some conclusions and future works.

2. General fuzzy relation equations

A complete linear lattice[1] $(L, \preceq)$ is the carrier considered throughout this paper, hence, the bottom and the top elements exist in L and are denoted as 0, 1, respectively. Given a set V, the ordering $\preceq$ in the lattice induces a partial order on the set of L-fuzzy subsets of V, that is, in the set L^V. This ordering is defined, for each pair of fuzzy subsets $S, S' \in L^V$, as $S \preceq S'$ if and only if $S(v) \preceq S'(v)$, for all $v \in V$. This ordering provides to L^V the structure of a complete lattice.

Moreover, the general residuated operator used to define the fuzzy relation equation is $\odot \colon L \times L \to L$, such that it is order preserving and there exists an operator $\to \colon L \times L \to L$, satisfying the following adjoint property with $\odot$

$$x \odot y \preceq z \quad \text{if and only if} \quad y \preceq x \to z \tag{1}$$

for each $x, y, z \in L$. Note that this property is equivalent to $\odot$ preserves supremums in the second argument; $x \odot \bigvee\{y \mid y \in Y\} = \bigvee\{x \odot y \mid y \in Y\}$, for all $Y \subseteq L$. Hence, very few properties are assumed. An important notion needed in this paper is the definition of *a cover*.[2]

Definition 1. Given an ordered set $(A, \preceq)$ and non-empty subsets $S_1, \ldots, S_n$ of A, an element $a \in A$ is *a cover of* $\{S_1, \ldots, S_n\}$, if for each $i \in \{1, \ldots, n\}$, there exists $s_i \in S_i$ such that $s_i \preceq a$. A cover $a \in A$ is called *minimal* if every element $d \in A$ satisfying $d \prec a$, is not a cover of $\{S_1, \ldots, S_n\}$.

Note that when $(A, \preceq)$ is a complete lattice, minimal covers always exist in A. Given the pair $(\odot, \to)$, a fuzzy relation equation in the environment of this paper is the equation:

$$R \circ X = T, \tag{2}$$

where $R \colon U \times V \to L$, $T \colon U \times W \to L$ are given finite L-fuzzy relations and $X \colon V \times W \to L$ is unknown, which can be expressed by matrices; and $R \circ X$ is defined, for each $u \in U$, $w \in W$, as

$$(R \circ X)\langle u, w\rangle = \bigvee\{R\langle u, v\rangle \odot X\langle v, w\rangle \mid v \in V\}$$

Therefore, this is an L-fuzzy relation which can also be written as a matrix.

It is well known that the fuzzy relation equation (2) has a solution if and only if

$$(R \Rightarrow T)\langle v, w\rangle = \bigwedge\{R\langle u, v\rangle \to T\langle u, w\rangle \mid u \in U\}$$

is a solution and, in that case, it is the greatest solution, see [22,26]. This result is constructive in the sense that, whenever solutions are known to exist, we can always compute at least one solution. The greatest solution is also a starting point in efforts to find and construct minimal solutions.

[1] See [7] for a detailed definition of this structure.

[2] The definition of cover (Definition 1) is different with [18,16] and [17]. Moreover, our approach is more general and it does not depend on the special type of the composition '∘' of fuzzy relations.

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