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Yoneda completeness and flat completeness of ordered fuzzy sets ☆

Wei Li, Hongliang Lai, Dexue Zhang *

School of Mathematics, Sichuan University, Chengdu 610064, China

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Abstract

This paper studies Yoneda completeness and flat completeness of ordered fuzzy sets valued in the quantale obtained by endowing the unit interval with a continuous triangular norm. Both of these notions are natural extension of directed completeness in order theory to the fuzzy setting. Yoneda completeness requires every forward Cauchy net converges (has a Yoneda limit), while flat completeness requires every flat weight (a counterpart of ideals in partially ordered sets) has a supremum. It is proved that flat completeness implies Yoneda completeness, but, the converse implication holds only in the case that the related triangular norm is either isomorphic to the Łukasiewicz t-norm or to the product t-norm.

Keywords: Category theory; Continuous t-norm; Quantale; Quantaloid; Ordered fuzzy set; Forward Cauchy net; Yoneda limit; Yoneda complete; Flat weight; Flat complete

1. Introduction

A partially ordered set P is directed complete if each directed subset of P has a supremum. Directed completeness is of fundamental importance in the theory of partial orders [14], so, it is tempting to establish a counterpart of this notion in the fuzzy setting. In this introduction, we explain the problems we encounter in this process, summarize briefly what has been done in the literature and what will be done in this paper.

Let $\mathcal{Q} = (Q, \&, 1)$ be a quantale. There are two kinds of fuzzy orders valued in $\mathcal{Q}$: the first is ($\mathcal{Q}$-valued) fuzzy orders on crisp sets; the second is orders on ($\mathcal{Q}$-valued) fuzzy sets. A crisp set together with a fuzzy order (valued in $\mathcal{Q}$) is, from the point of view of category theory, a category enriched over $\mathcal{Q}$ (considered as a one-object monoidal biclosed category), with generalized metric spaces in the sense of Lawvere [32] and fuzzy orders in the sense of Zadeh [49] as prominent examples. A fuzzy set (valued in $\mathcal{Q}$) equipped with an order is a category enriched in the quantaloid $\mathsf{D}(\mathcal{Q})$ of diagonals in $\mathcal{Q}$ [20,21,35,42], with localic sheaves [4,47], Ω-posets [7], and generalized partial metric spaces [21,27,33,35] as prototypes. So, both kinds of fuzzy orders are enriched categories, with the first being over a quantale and the second over a quantaloid. The reader is referred to [42] for a nice introduction to the relationship between fuzzy

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* Corresponding author. Tel: +86 13558678979.
E-mail addresses: mathli@foxmail.com (W. Li), hllai@scu.edu.cn (H. Lai), dxzhang@scu.edu.cn (D. Zhang).

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orders and quantaloid-enriched categories, see also [20,21,35]. It should be noted that the category of $\mathcal{Q}$-categories (i.e., crisp sets together with $\mathcal{Q}$-valued orders) can be identified with a full subcategory of $\mathsf{D}(\mathcal{Q})$-categories (i.e., ordered fuzzy sets valued in $\mathcal{Q}$). So, the study of fuzzy orders on crisp sets is a special case of that of orders on fuzzy sets. Fuzzy orders on crisp sets have received wide attention (to name a few, [2,3,6,19,29,30,39,45,46,48]), but, the general theory of orders on fuzzy sets is still at its beginning steps, many things remain to be unveiled.

Directed completeness of $\mathcal{Q}$-categories has already received wide attention. The aim of this paper is to investigate directed completeness of ordered fuzzy sets (or, $\mathsf{D}(\mathcal{Q})$-categories) in the case that $\mathcal{Q}$ is the unit interval [0, 1] coupled with a continuous t-norm &. These quantales play an important role in fuzzy set theory, for instance, the BL-logic [16] is a logic based on continuous t-norms.

In order to explain what has been done in the literature and what will be done in this paper, we recall two equivalent characterizations of directed completeness first. Let P be a partially ordered set. A subset of P is called an ideal if it is a directed and a lower set. A net $\{x_i\}$ in P is eventually monotone if there is some i such that $x_j \leq x_k$ whenever $i \leq j \leq k$. An element $y \in P$ is an eventual upper bound of a net $\{x_i\}$ if there is some i such that $x_j \leq y$ whenever $i \leq j$. For a partially ordered set P, it is clear that the following are equivalent:

- P is directed complete.
- Each ideal of P has a supremum.
- For each eventually monotone net $\{x_i\}$ in P, there is some $x \in P$ such that for all $y \in P$, $x \leq y$ if and only if y is an eventual upper bound of $\{x_i\}$. Said differently, each eventually monotone net has a least eventual upper bound.

Both the approach of ideals (i.e., each ideal has a supremum) and the approach of nets (i.e., each eventually monotone net has a least eventual upper bound) to directed completeness have been extended to $\mathcal{Q}$-categories. For the approach of nets, forward Cauchy nets (a $\mathcal{Q}$-version of eventually monotone nets) have been introduced, resulting in the notion of Yoneda complete $\mathcal{Q}$-categories (a.k.a. liminf complete $\mathcal{Q}$-categories), see e.g. [6,12,15,26,31,44,46]. For the approach of ideals, certain classes of weights in $\mathcal{Q}$-categories have been proposed as $\mathcal{Q}$-versions of ideals, see e.g. [10–12,25,30,40,44,48]. The approach via ideals is in fact an instance of the theory of Φ-cocompleteness for enriched categories [1,22,23]. It should be stressed that the situation with $\mathcal{Q}$-categories is much more complicated than the classic case. To see this, we list two facts here. The first, there lacks a “standard” choice of weights that can be treated as a counterpart of ideals in partially ordered sets. Instead, different classes of weights have been proposed in the literature for different kinds of quantales. For instance, $\mathcal{V}$-ideals for $\mathcal{V}$-continuity spaces [10–12], flat weights for generalized metric spaces [44], etc. The second, though the two approaches are equivalent in the classic case, their relationship is not clear in the fuzzy setting, see e.g. [19,40]. However, there have been some interesting results in this regard. For instance, for $\mathcal{V}$-continuity spaces, Yoneda completeness is equivalent to that each $\mathcal{V}$-ideal has a supremum [11,12]; for generalized metric spaces, Yoneda completeness is equivalent to that each flat weight has a supremum [44].

In this paper, both approaches to directed completeness will be extended to ordered fuzzy sets in the case that $\mathcal{Q}$ is the quantale obtained by endowing the unit interval [0, 1] with a continuous t-norm &. For the approach of ideals, flat completeness, that requires every flat weight has a supremum, is considered; for the approach of nets, Yoneda completeness, that requires every forward Cauchy net has a Yoneda limit, is considered. The focus is on the relationship between flat completeness and Yoneda completeness. It is shown that flat completeness always implies Yoneda completeness, but, the converse implication holds only when the t-norm & is either isomorphic to the Łukasiewicz t-norm or to the product t-norm. These results exhibit a deep connection between properties of ordered fuzzy sets and the structure of $\mathcal{Q}$ — the table of truth-values.

The contents are arranged as follows. Section 2 recalls some basic ideas about quantales, continuous t-norms, and quantaloids. Section 3 recalls the notion of ordered fuzzy sets valued in a quantale $\mathcal{Q}$, with emphasis on the fact that they are categories enriched over a quantaloid constructed from $\mathcal{Q}$. Section 4 introduces the notions of flat weights and flat completeness for ordered fuzzy sets. These notions make sense for categories enriched in any quantaloid. Section 5 introduces the concepts of forward Cauchy nets and Yoneda completeness for ordered fuzzy sets and presents some of their basic properties. Section 6 shows that if & is a continuous t-norm, then flat completeness implies Yoneda completeness for ordered fuzzy sets valued in the quantale $([0,1], \&, 1)$. Section 7 proves that if & is a continuous t-norm, then each Yoneda complete ordered fuzzy set (valued in the quantale $([0,1], \&, 1)$) with an isolated element

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