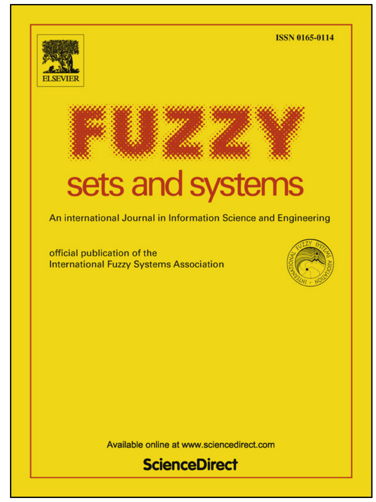

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PII: S0165-0114(16)30118-X
DOI: http://dx.doi.org/10.1016/j.fss.2016.04.013
Reference: FSS 7031

To appear in: *Fuzzy Sets and Systems*

Received date: 24 February 2015
Revised date: 21 April 2016
Accepted date: 23 April 2016

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Abstract

In the context of quantaloid-enriched categories, we rely essentially on the classifying property of presheaf categories to give a conceptual proof of a theorem due to Höhle: the double power monad and the composite power monad, on the category of quantaloid-enriched categories, are the same. Via the theory of distributive laws, we identify the algebras of this monad to be the completely codistributive complete categories, and the homomorphisms between such algebras are the bicontinuous functors. With these results we hope to contribute to the further development of a theory of Q-valued preorders (in the sense of Pu and Zhang).

Keywords: Category theory, Fuzzy preorders, Non-classical logics

1. Introduction

If $P = (P, \leq)$ is an ordered set, then its downclosed subsets form a sup-lattice $(\mathsf{Dwn}(P), \subseteq)$, and the order-preserving inclusion $P \longrightarrow \mathsf{Dwn}(P)\colon x \mapsto \downarrow x$ has a left adjoint if and only if P has all suprema. Dually, taking the upclosed subsets of P produces an inf-lattice $(\mathsf{Up}(P), \supseteq)$ (note that upsets are ordered by containment, whereas downsets are ordered by inclusion), and the order-preserving inclusion $P \longrightarrow \mathsf{Up}(P)\colon x \mapsto \uparrow x$ has a right adjoint if and only if P has all infima. Of course, P is a sup-lattice if and only if it is an inf-lattice, and then it is said to be a 'complete lattice'.

These two object correspondences can be made functorial in several ways, and the resulting functors interact in at least two ways. For starters, the inverse image of an order-preserving function $f\colon P \longrightarrow Q$ is a new order-preserving function $f^{-1}\colon \mathsf{Dwn}(Q) \longrightarrow \mathsf{Dwn}(P)$. This action on objects and morphisms defines a 2-functor on the locally ordered category Ord of ordered sets which reverses arrows and local order; for the sake of this introduction, let us write it as $\mathcal{L}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}^{\mathsf{coop}}$. It then so happens that this is a left 2-adjoint, and that the action of its right 2-adjoint $\mathcal{R}\colon \mathsf{Ord}^{\mathsf{coop}} \longrightarrow \mathsf{Ord}$ on objects is $Q \mapsto \mathsf{Up}(Q)$. As a result, the induced 2-monad $\mathcal{T} := \mathcal{R}\mathcal{L}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ acts on objects as $P \mapsto \mathsf{Up}(\mathsf{Dwn}(P))$: it is the **double power monad** on Ord.

On the other hand it is well-known that the locally ordered category Sup of sup-lattices and sup-morphisms is included in Ord by a forgetful 2-functor $\mathcal{U}\colon \mathsf{Sup} \longrightarrow \mathsf{Ord}$, right 2-adjoint to an $\mathcal{F}\colon \mathsf{Ord} \longrightarrow \mathsf{Sup}$ whose action on objects is $P \mapsto \mathsf{Dwn}(P)$; a 2-monad $\mathsf{Dwn} = \mathcal{U}\mathcal{F}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ results, and its action on objects is $P \mapsto \mathsf{Dwn}(P)$. In a similar manner, because the forgetful 2-functor $\mathcal{V}\colon \mathsf{Inf} \longrightarrow \mathsf{Ord}$ admits a left 2-adjoint $\mathcal{G}\colon \mathsf{Ord} \longrightarrow \mathsf{Inf}$, their composition produces a 2-monad $\mathsf{Up} = \mathcal{V}\mathcal{G}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$, whose action on objects is $Q \mapsto \mathsf{Up}(Q)$. Now it turns out that the composition of these 2-monads, $\mathcal{S} := \mathsf{Up}\mathsf{Dwn}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$, is again a 2-monad, and its action on objects is thus $P \mapsto \mathsf{Up}(\mathsf{Dwn}(P))$: it is the **composite power monad** on Ord.

In this note we show how **the double power monad and the composite power monad are the same**. We prove this in the generality of **quantaloid-enriched categories**, of which not only ordered sets but also metric spaces [Lawvere, 1973], partial metric spaces [Höhle and Kubiak, 2011; Stubbe, 2014],

[☆]Submitted to Fuzzy Sets and Systems: February 19, 2015; this revised version: April 21, 2016.
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