ARTICLE IN PRESS
FSS:7054
Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems ••• (••••) •••–•••
FUZZY sets and systems
www.elsevier.com/locate/fss

A fuzzy approach to quantum logical computation

C. Bertini [a], R. Leporini [b,*]

[a] *Dipartimento di Scienze Aziendali, Economiche e Metodi Quantitativi, Università di Bergamo, via dei Caniana 2, I-24127 Bergamo, Italy*
[b] *Dipartimento di Ingegneria Gestionale, dell'Informazione e della Produzione, Università di Bergamo, viale Marconi 5, I-24044 Dalmine (BG), Italy*

Received 29 November 2015; received in revised form 8 June 2016; accepted 9 June 2016

Abstract

The theory of logical gates in quantum computation has inspired the development of new forms of quantum logic, called quantum computational logics. The basic semantic idea is the following: the meaning of a formula is identified with a quantum information quantity, represented by a density operator, whose dimension depends on the logical complexity of the formula. At the same time, the logical connectives are interpreted as operations defined in terms of quantum gates. In this framework, some possible relations between fuzzy representations based on continuous t-norms for quantum gates and the probabilistic behavior of quantum computational finite-valued connectives are investigated.

Keywords: Fuzzy connectives; Non-classical logics; Quantum gates

1. Introduction

The mathematical formalism of quantum theory has inspired the development of different forms of non-classical logics, called *quantum logics*. In many cases the semantic characterizations of these logics are based on special classes of algebraic structures defined in a Hilbert-space environment. Interesting generalizations of quantum logic introduced by Birkhoff and von Neumann are the so called *unsharp (or fuzzy) quantum logics* that can be semantically characterized by referring to different classes of algebraic structures whose support is the set of all *effects* of a Hilbert space [6].

A different approach to quantum logic has been developed in the framework of *quantum computational logics*, inspired by the theory of quantum computation [8,9,2]. While sharp and unsharp quantum logics refer to possible structures of physical events, the basic objects of quantum computational logics are *pieces of quantum information*: possible states of quantum systems that can store the information in question. The simplest piece of quantum information is a *qubit*: a unit-vector of the Hilbert space $\mathbb{C}^2$ that can be represented as a superposition $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. The two elements of the canonical basis of $\mathbb{C}^2$, $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$, represent the classical bits or, equivalently,

* Corresponding author.
E-mail addresses: cesarino.bertini@unibg.it (C. Bertini), roberto.leporini@unibg.it (R. Leporini).

http://dx.doi.org/10.1016/j.fss.2016.06.004
0165-0114/

Please cite this article in press as: C. Bertini, R. Leporini, A fuzzy approach to quantum logical computation, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.06.004

the two classical truth-values. It is interesting to consider a "many-valued generalization" of qubits, represented by *qudits*: unit-vectors living in a space $\mathbb{C}^d$, where $d \geq 2$.

The aim of this paper is to study a probabilistic type representation for logical gates based on product t-norm, Łukasiewicz sum and some many valued connectives in the framework of quantum computation with density operators. Any formula of the language gives rise to a quantum circuit that transforms the density operator associated to the formula into the density operator associated the atomic subformulas in a reversible way [9]. One of the advantages of this probabilistic type representation is that we can deal with such circuits as expressions in an algebraic environment (as in the case of Boolean algebra to describe digital circuits).

The paper is organized as follows. In Sections 2–3, we introduce basic notions of quantum computational logics and recall some gates that play a special role from the logical point of view and some interesting relations between these gates and the probability function p. In Section 4, we introduce matrix basis decompositions for density matrices associated to states of d-dimensional quantum systems and describe a state tomography scheme. In Section 5, we show some interesting relations between the logical gates and continuous t-norms by probability values. Finally, in Sections 6–7, we describe the capacity for some holistic connectives of characterizing entanglement of formation both for isotropic states and for Werner states.

2. The basic notions

Let us first recall some basic definitions. As is well known, the general mathematical environment for quantum computation is the Hilbert space $\mathcal{H}^{(n)} := \underbrace{\mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d}_{n\text{-times}}$ (n-fold tensor product where $n \geq 1$ and $d \geq 2$). The canonical orthonormal basis $\mathcal{B}^{(n)}$ of $\mathcal{H}^{(n)}$ is defined as follows:

$$\mathcal{B}^{(n)} = \left\{ |x_1, \ldots, x_n\rangle : x_1 \in \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, 1\right\}, \ldots, x_n \in \left\{0, \frac{1}{d-1}, \frac{2}{d-1}, \ldots, 1\right\}\right\},$$

where $|0\rangle = (1, 0, \ldots, 0)$, $|\frac{1}{d-1}\rangle = (0, 1, 0, \ldots, 0)$, $|\frac{2}{d-1}\rangle = (0, 0, 1, 0, \ldots, 0)$, $\ldots$, $|1\rangle = (0, \ldots, 0, 1)$, while $|x_1, \ldots, x_n\rangle$ is an abbreviation for the tensor product $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$.

Any piece of quantum information is represented by a density operator ρ of a space $\mathcal{H}^{(n)}$. A *quregister* (or quregister-state) is represented by a unit-vector $|\psi\rangle$ (which is a pure state) of a space $\mathcal{H}^{(n)}$ or, equivalently, by the corresponding density operator $\mathtt{P}_{|\psi\rangle}$ (the projection-operator that projects over the closed subspace determined by $|\psi\rangle$). Following a standard convention, we assume that $\mathtt{P}_{|1\rangle}$ represents the truth-value *Truth*, $\mathtt{P}_{|0\rangle}$ represents the truth-value *Falsity* and $\mathtt{P}_{|\frac{j}{d-1}\rangle}$ represent *intermediate* truth-values (where $0 < j < d-1$).

In this framework, one can define the projections that represent the *Truth*, the *Falsity* and intermediate properties in any space $\mathcal{H}^{(n)}$. A *truth-value projection* of $\mathcal{H}^{(n)}$ is a projection $P^{(n)}_{\frac{j}{d-1}}$ whose range is the closed subspace spanned by the set of all quregisters ending with $\frac{j}{d-1}$ of $\mathcal{H}^{(n)}$, where $0 \leq j \leq d-1$.

Accordingly, by applying the Born rule, one can now define the probability that ρ is true, false and an intermediate truth-value in $\mathcal{H}^{(n)}$:

$$\mathrm{p}_{\frac{j}{d-1}}(\rho) = \mathrm{tr}\left(\rho\, P^{(n)}_{\frac{j}{d-1}}\right),$$

where $0 \leq j \leq d-1$ and tr is the trace-functional.

From an intuitive point of view, $\mathrm{p}_{\frac{j}{d-1}}(\rho)$ represents the probability that the information stocked by the density operator ρ is the truth-value $\frac{j}{d-1}$.

One can now define the probability for any density operator ρ of $\mathcal{H}^{(n)}$ as the weighted mean of the truth-values.

Definition 1. The probability of a density operator.

$$\mathrm{p}(\rho) = \frac{1}{d-1}\sum_{j=1}^{d-1} j\, \mathrm{p}_{\frac{j}{d-1}}(\rho)$$

Please cite this article in press as: C. Bertini, R. Leporini, A fuzzy approach to quantum logical computation, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.06.004