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Products of lattice-valued fuzzy transition systems and induced fuzzy transformation semigroups

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Abstract

In this study, several products of fuzzy transformation semigroups are generalized to the case where their truth structure is a complete lattice. We show that the direct product of two such fuzzy transformation semigroups is again a fuzzy transformation semigroup if and only if the lattice is distributive. In addition, after extending the cascade product, we introduce a novel way of building a fuzzy transformation semigroup starting from a unique automaton. The particular case of the lattice comprising all the closed subintervals contained in [0, 1] is analyzed.

Keywords: Fuzzy transition system; Interval-valued fuzzy set; Products of automata; Transformation semigroup

1. Introduction

Fuzzy finite state machines and fuzzy transformation semigroups, as well as the relationships between them, have been studied widely (see [16]). In recent studies ([3,4], and [13]), fuzzy finite state machines have been referred to as fuzzy transition systems, especially when applied in areas such as discrete event systems, process calculi, modal logic, and model checking.

The basic idea of their formulation is that unlike the crisp case, they can switch from one state to another one to a certain truth degree between 0 and 1. However, the truth structure of these automata has frequently been extended from the real interval [0, 1] to a more general complete lattice (see [17]). For instance, Belohlavek [1] dealt with fuzzy automata with membership values in a complete lattice. Subsequently, Li [14] focused on fuzzy automata with membership values in a lattice-ordered monoid.

Following Malik and Morderson [15], Droste et al. [5,6] studied the direct product of weighted finite transition systems and the transformation semigroups obtained from them in a very particular case. They only considered weights belonging to naturally ordered semirings, which are always distributive.

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In a complementary manner, the cascade product of semiring-weighted transition systems was studied by [6] in order to obtain a covering of any of these automata by the cascade product of two simpler automata obtained as quotients of the original. However, due to the poor behavior of this product when the set of inputs is extended from letters to words, the cascade product of fuzzy transformation semigroups was not analyzed.

In the present study, we analyze the general direct product for L-fuzzy transition systems in the more general case of a complete lattice L that is not necessarily distributive. We show that the direct product of the L-fuzzy transformation semigroups obtained from the transition systems in the usual manner is also an L-fuzzy transformation semigroup if and only if the distributivity holds (Theorem 3.4).

In addition, we study the cascade product of the L-fuzzy transformation semigroups obtained from two L-fuzzy transition systems in the case of a complete distributive lattice L. We show that this cascade product is an L-fuzzy transformation semigroup if it is strongly separable.

This case is sufficient to allow us to introduce the notion of an induced transformation semigroup (from a group contained within it). This new concept allows the construction of a new L-fuzzy transformation semigroup by using the cascade product, but starting from a unique automaton instead of the two automata required in the classical construction. The new L-fuzzy transformation semigroup comprises the union of several copies of the original, which provides a model of some automata working in parallel. We show that the construction based on the induction of an L-fuzzy transformation semigroup satisfies *good* properties such as transitivity, or it preserves the faithfulness of the original automaton.

The particular case where L is the set of all the closed intervals contained in $[0, 1]$ is analyzed due to several reasons. This lattice is one of the most suitable for modeling some situations of uncertainty that arise in practice (see [2]). In addition, the information provided by the transition functions can be retrieved by using aggregation functions in this case.

The remainder of this paper is organized as follows. Section 2 provides the main definitions and results regarding lattice-valued fuzzy finite state automata. In Sections 3 and 4, we study the direct products and cascade products between lattice-valued fuzzy transition systems, respectively. In Section 5, we introduce the induced transformation semigroup. In Section 6, we consider the particular case of interval-valued automata. Finally, we give our conclusions.

2. Preliminaries

Throughout this section, $(L, \leq_L)$ is a complete lattice, i.e., a partially ordered set for which every subset has an infimum (greatest lower bound) and a supremum (least upper bound). We denote 0_L as the least element in the whole set L and 1_L as the greatest element of L.

For $(L, \leq_L)$, we only consider the t-norm given by the meet denoted by $\wedge$ and the t-conorm given by the join denoted by $\vee$. Note that both norms R satisfy the following *local finiteness property* (see [8–10]).

For each finite subset $Y \subseteq L$,

$$\{R(a_1, \ldots, a_n) \mid a_1 \in Y, \ldots, a_n \in Y;\ n \geq 1\} \text{ is finite.}$$

Remark 2.1. If L is the real interval $[0, 1]$, then the t-norm given by the product does not satisfy the local finiteness property. Indeed, if $Y = \{1/2\}$, then we obtain

$$\{a_1 \cdot a_2 \cdots \cdot a_n \mid a_1 \in Y,\ a_2 \in Y, \ldots, a_n \in Y;\ n \geq 1\} = \{1/2^n;\ n \geq 1\},$$

which is clearly infinite.

Remark 2.2. The t-norm given by the meet is not necessarily distributive with respect to the t-conorm given by the join (see [11]). Indeed, the following distributive properties are equivalent for any complete lattice $(L, \leq_L)$:

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in L$.
(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.

A complete lattice $(L, \leq_L)$ where one (and thus both) of the previous distributive properties holds is called *a complete distributive lattice*.