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Dynamics and topological entropy for Zadeh's extension of a compact system ☆

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Abstract

In this study, we consider the dynamics of a fuzzy system extended by a crisp compact system, which has a mixing regular periodic decomposition. First, we describe the complete dynamics of Zadeh's extension of a crisp compact system, which has a mixing regular periodic decomposition, by splitting the space of fuzzy numbers on the compact space. Next, using the results obtained, we prove that the topological entropies of the crisp system and its Zadeh's extension to the space of fuzzy numbers are the same.

Keywords: Fuzzy discrete dynamical system; Limit set; Measure-theoretical entropy; Mixing regular periodic decomposition; Topological entropy

1. Introduction

Let X be a compact metric space and $f : X \to X$ be a continuous map. This map f is extended in a natural manner to a continuous self-map $\hat{f}$ defined on the space of fuzzy numbers on X by Zadeh's extension principle ([19]). Therefore, we can consider a new discrete dynamical system $(\mathcal{F}(X), \hat{f})$ extended from the discrete dynamical system (X, f), where $\mathcal{F}(X)$ denotes the space of fuzzy numbers on X. In this context, the following natural question arises.

What are the relationships between the dynamical properties of the systems mentioned above?

In recent years, several studies have considered this problem ([7,8,10–12,17]). In [8], by using the variational principle, it was proved that the topological entropies of an interval map and its Zadeh's extension to the space of fuzzy numbers are the same. They also suggested that their results could be generalized further to more general dynamical systems (e.g., a tree or circle). Moreover, they emphasized that their results are not valid for general discrete dynamical systems that are defined on compact metric space (e.g., see [1]), so this deficiency should be studied.

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In this study, we investigate some aspects of the problems suggested in [8], i.e., the relationships between the topological entropies of a compact system with a mixing regular periodic decomposition and its Zadeh's extension to the space of fuzzy numbers. We show that the topological entropies of the two systems are the same.

Most studies of fuzzy dynamical systems have considered the relationships with set-valued dynamical systems, and the results obtained in the field of set-valued dynamical systems are important for fuzzy systems. In [13], the complete dynamics of the induced map were described by a transitive graph map and as corollaries, it was shown that the induced map described by a graph map is never transitive, and that the topological entropies of a transitive graph map and its induced hyperspace map are the same.

We describe the complete dynamics of the Zadeh's extension of a compact system (X, f), which has a mixing regular periodic decomposition. By splitting the space of fuzzy numbers $\mathcal{F}(X)$ into three disjoint subsets, i.e., $\mathcal{F}_0(X)$, $\mathcal{F}_1(X)$, and $\mathcal{F}_2(X)$, we give a complete description of the behaviors of the orbits of points in the fuzzy system. In particular, we prove that the ω-limit set of a point in $\mathcal{F}_1(X)$ is empty and that the ω-limit set of a point in $\mathcal{F}_i(X)$ $(i = 0, 2)$ is nonempty and contained in the same $\mathcal{F}_i(X)$, and we also present the complete compositions of ω-limit sets. As a corollary, we give the complete compositions of ω-limit sets in the fuzzy system extended by a transitive graph map.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic definitions and previous results, which are required in this study. In Section 3, we give the characteristic properties of ω-limit sets for the Zadeh's extension of a compact system with a mixing regular periodic decomposition. Section 4 explains the relationship between the topological entropies of the two maps.

2. Preliminaries

Let $\mathbf{N}$ denote the set of all positive integers and $\mathbf{N_0} = \mathbf{N} \cup \{\mathbf{0}\}$. Let (X, d) be a metric space and $f : X \to X$ be a continuous map. A pair (X, f) is called a *discrete dynamical system*. We refer to a nonempty subset $I \subset X$ *invariant* as f (f*-invariant*) if $f(I) \subset I$, and if $f(I) = I$, we refer to I *strictly invariant* as f (*strictly* f*-invariant*). For a given $x \in X$, the sequence $(f^n(x))_{n \in \mathbf{N_0}}$ is called the *trajectory* of the point x and we define the *orbit* of the point x by the set $\{f^n(x) : n \in \mathbf{N_0}\}$, which is denoted as $\mathrm{Orb}(x, f)$. The ω*-limit set* of the point x with respect to the map f is the set of all limit points of the trajectory $(f^n(x))_{n \in \mathbf{N_0}}$, which is denoted as $\omega(x, f)$, and the ω-limit set $\omega(f)$ of the map f is the set $\bigcup_{x \in X} \omega(x, f)$. If a ω-limit set $\omega(x, f)$ is nonempty, then this ω-limit set $\omega(x, f)$ is closed and strictly f-invariant, and it also holds that

$$\omega(x, f) = \bigcup_{j=0}^{n-1} \omega(f^j(x), f^n), \tag{1}$$

for any $n \in \mathbf{N}$.

We say that x is the *periodic point* of f if $f^n(x) = x$ for some $n \in \mathbf{N}$ and the minimal $k \in \mathbf{N}$ that satisfies this condition is called a *period* of x, where the set of all periodic points of f is denoted as $P(f)$.

A map f is *transitive* if for any two nonempty open sets $U, V \subset X$, an $n \in \mathbf{N}$ exists such that $f^n(U) \cap V \neq \emptyset$, and if f^n is transitive for any $n \in \mathbf{N}$, then we call f *totally transitive*. We say that f is *mixing* if for any two open sets $U, V \subset X$, an $N \in \mathbf{N}$ exists such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

Let (X, f) and (Y, g) be discrete dynamical systems. If a continuous surjective map $\pi : X \to Y$ exists such that $\pi \circ f = g \circ \pi$, then we say that π *topologically semiconjugates* f to g (or f is *topologically semiconjugate* to g); π is a *topological semiconjugacy* from X to Y; and g is a *factor* of f (via π). If the map π is a homoeomorphism, then we call π a *topological conjugacy* between f and g, and in this case, we say that f and g (or (X, f) and (Y, g)) are *topologically conjugate* (via π). If g is a factor of f via π and an $M \in \mathbf{N}$ exists such that $\#\{\pi^{-1}(y)\} \leq M$ for every $y \in Y$ ($\#A$ is the cardinality of the set A), then we say that π is a *uniformly finite-to-one* semiconjugacy from f to g.

Let G be a connected compact Hausdorff space. If a finite collection of subspaces I_i $(i \in \{1, \cdots, s\})$ exists such that $G = \bigcup I_i$, each I_i is homeomorphic to a compact interval, and each intersection $I_i \cap I_j$ for $i \neq j$ is finite, then we call G a (*topological*) *graph*. In addition, a continuous map $f : G \to G$ is called a *graph map*.

2.1. Regular periodic decompositions

Let us recall the notion of regular periodic decomposition introduced in [3,4]

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