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Approximation of fuzzy numbers using the convolution method

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Abstract

In this study, we consider the use of the convolution method for constructing approximations comprising fuzzy number sequences with useful properties for a general fuzzy number. We show that this convolution method can generate differentiable approximations in finite steps for fuzzy numbers with finite non-differentiable points. In a previous study, this convolution method was only used for constructing differentiable approximations of continuous fuzzy numbers, the possible non-differentiable points of which were the two endpoints of the 1-cut. The construction of smoothers is a key step in the process for producing approximations. We also show that if appropriate smoothers are selected, then we can use the convolution method to provide approximations that are differentiable, Lipschitz, and that also preserve the core.

Keywords: Approximation; Convolution; Differentiable; Fuzzy number; Supremum metric

1. Instructions

The approximations of fuzzy numbers has attracted much attention from researchers. In general, these studies can be grouped into two classes. One class uses a given shape fuzzy number to approximate the original fuzzy number. There have been many important studies in this area, including the following. Chanas [6] and Grzegorzewski [15] independently presented the interval approximations. Ma et al. [22] presented the symmetric triangular approximations. Abbasbandy and Asady [1] presented the trapezoidal approximations. Grzegorzewski and Mrówka [16,17] presented the trapezoidal approximations that preserve the expected interval. Zeng and Li [34] presented the weighted triangular approximations. Nasibov and Peker [23] presented the semi-trapezoidal approximations, which were improved by Ban [2,3]. Yeh [30] presented the weighted semi-trapezoidal approximations. Yeh and Chu [31] presented a unified method for solving LR-type approximation problems without constraints according to the weighted L_2-metric. Coroianu [11] discussed how to find the best Lipschitz constant for the trapezoidal approximation operator while preserving the value and ambiguity. Ban and Coroianu [4] proposed simpler methods for computing the parametric approximation of

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a fuzzy number while preserving some important characteristics. Studies of this class of fuzzy number approximations have provided various methods for approximating an arbitrary fuzzy number according to some metrics based on a special type of fuzzy number, which is much more convenient to calculate. In addition, this approach finds the fuzzy number with the minimal distance to the original fuzzy number among all the given types of fuzzy numbers, so it minimizes the loss of information to a certain extent.

However, in many cases, when the distance between the approximated fuzzy number and original fuzzy number is smaller, the effect appears to be better. Thus, it is also important to consider the problem of whether we can approximate an arbitrary fuzzy number well using fuzzy numbers with useful properties, e.g., continuous and differentiable. This is the subject of another class of fuzzy number approximation methods, which consider how to construct a sequence of fuzzy numbers with some properties that approximate a general fuzzy number. There have also been many important studies in this area, including the following. Colling and Kloeden [8] used a sequence of continuous fuzzy numbers to approximate an arbitrary fuzzy number. Coroianu et al. [9,10] constructed approximations that comprised a sequence of fuzzy numbers by using the F-transform and the max-product Bernstein operators. Román-Flores et al. [24] noted that sequences of Lipschitzian fuzzy numbers can approximate any fuzzy numbers. To demonstrate this fact, they presented a method based on the convolution of two fuzzy numbers to construct approximations for fuzzy numbers. For convenience, we refer to this as the convolution method in the sequel. This convolution method began with a study by Seeger and Volle [25].

Differentiable fuzzy numbers play an important role in the implementation of fuzzy intelligent systems and their applications (see [7,18,29]). For instance, the fuzzy numbers need to be differentiable to use the well-known gradient descent algorithm. Thus, it is important to know whether we can use a sequence of differentiable fuzzy numbers to approximate a general fuzzy number. Chalco-Cano et al. [32,33] used the convolution method to construct sequences of differentiable fuzzy numbers to approximate a type of non-differentiable fuzzy number under the supremum metric. The convergence induced by the supremum metric is stronger than that induced by the L_p-metric, sendograph metric, endograph metric, and level-convergence (see [12,19–21,28]), so it follows that the constructed approximation is also an approximation of the original fuzzy number under the convergences mentioned above. This method is easy to implement because it operates on level-cut sets of the fuzzy numbers. In the sequel, a differentiable fuzzy number is also called a smooth fuzzy number, a non-differentiable fuzzy number is called a non-smooth fuzzy number, and an approximation comprising a sequence of differentiable fuzzy numbers is called a smooth approximation.

To construct smooth approximations for a type of non-smooth fuzzy number, Chalco-Cano et al. [32,33] demonstrated that the convolution transform can be used to smooth this type of fuzzy numbers, i.e. it can transform a non-smooth fuzzy number of this type into a smooth fuzzy number. In fact, they constructed a class of "smoothers" as fuzzy numbers that satisfy some conditions. Given a non-smooth fuzzy number of this type, we can obtain a smooth fuzzy number via convolution of the original fuzzy number and the smoother. The construction of smoothers is an important step in the construction of approximations. The distance between the smooth fuzzy number and the original fuzzy number can be controlled by the smoother. Thus, by choosing an appropriate smoother, we can obtain a smooth fuzzy number such that the distance between it and the original fuzzy number is less than an arbitrarily small positive number given in advance. Thus, we can produce a sequence of smooth fuzzy numbers, which comprise a smooth approximation of the original fuzzy numbers.

However, in the previous study, only a particular type of fuzzy number could be smoothed by the convolution method. Hence, only this type of fuzzy number can be smoothly approximated using the convolution method. This type of fuzzy number has at most two possible non-differentiable points, which are the endpoints of the 1-cut. By contrast, an arbitrary fuzzy number may have other non-differentiable points, or even non-continuous points. Thus, it is natural to ask whether we can use the convolution method to smooth a general fuzzy number and whether we can obtain a smooth approximation of the original fuzzy number.

In this study, we address these questions, where we first discuss the properties of fuzzy numbers and the convolution of fuzzy numbers. Based on these discussions, we give partial positive answers to the questions posed above. The key is the construction of smoothers for a general fuzzy number so that it can be smoothed, which we perform step by step. First, we show how to construct smoothers for a subtype of continuous fuzzy numbers. Next, we investigate how to construct smoothers for continuous fuzzy numbers. Finally, we explore how to construct smoothers for an arbitrary fuzzy number so it can be transformed into a smooth fuzzy number. Based on these results, we show how to construct smooth approximations for fuzzy numbers with finite non-differentiable points. These types of fuzzy numbers are quite general in real-world applications. We also show that by choosing appropriate smoothers, the smooth approximations