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Approximations of fuzzy numbers by step type fuzzy numbers ☆

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Received 10 July 2015; received in revised form 6 January 2016; accepted 14 August 2016

Abstract

In this paper, the problems of approximations in membership functions space are studied. Firstly, the concept of simple fuzzy numbers is defined, which is a special kind of fuzzy numbers with simple structure, and give the expressions of the membership function of thus fuzzy numbers. Then, we prove that a fuzzy set of real number field R is a simple fuzzy number if and only if it is a normal step type fuzzy set, and obtain a result of that simple fuzzy number space is dense in the fuzzy number space with respect to some metric. And then, we give two theorems about approximations of fuzzy number membership functions by using simple fuzzy number membership functions. At last, we investigate the properties of approximation operators given by us.

Keywords: Approximations; Membership functions; Fuzzy numbers; Simple fuzzy numbers

1. Introduction

It is known that approximation theory is an important research field in mathematics, and has strong application background. In this research field, many important results have been made. Recently, there is still a lot of work about function approximations. For example, in [6], Drugowitsch and Barry introduced parts of a formal framework that aims at studying learning classifier systems, concerning how a set of classifiers approximate a function separately and in combination; In [17], Nikolaev established a relation between bijective functions and renormalization group transformations and find their renormalization group invariants, and proposed several improved approximations (compared with the power series expansion) based on this relation; In [18], Pekarskii dealt with the order of best rational approximations to function z^α in a domain with zero external angle and vertex at the point $z = 0$; In [14], Jafarov investigated the approximation properties of trigonometric polynomials in rearrangement invariant quasi Banach function spaces; In [4], Darabi and Itskov presented an analytical method based on the Padé technique and the multiple point interpolation for the inverse Langevin function.

About fuzzy numbers, the concept was proposed by Chang and Zadeh to study the properties of probability functions in [2]. With the development of theories and applications of fuzzy number, this concept becomes more and more

☆ This work is supported partially by the Nature Science Foundation of China (No. 61273077).
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http://dx.doi.org/10.1016/j.fss.2016.08.003
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Please cite this article in press as: G. Wang, J. Li, Approximations of fuzzy numbers by step type fuzzy numbers, Fuzzy Sets Syst. (2016), http://dx.doi.org/10.1016/j.fss.2016.08.003

important. The problem of approximations of membership functions of fuzzy numbers is also an important research area [12], and this topic has been extensively studied in the last decade. For example, in [19–22], Yeh studied the problems of trapezoidal, triangular, weighted trapezoidal, weighted triangular and weighted semi-trapezoidal approximations of fuzzy numbers, and got some important results; In [1], Chanas proposed an approximation interval of a fuzzy number which keeps the same width of the interval and the fuzzy number, and satisfies that the Hamming distance between this interval and the approximated number is minimal; In [10], Grzegorzewski proposed a new interval approximation operator, which is the best one with respect to a certain measure of distance between fuzzy numbers; In [11], Grzegorzewski and Mrówka gave a new nearest trapezoidal approximation operator preserving expected interval; In [3], Coroianu, Gagolewski and Grzegorzewski obtained approximations which are simple enough and flexible to reconstruct the input fuzzy concepts under study by using 1-knot fuzzy numbers. In fact, whether interval approximations of fuzzy numbers or triangular approximations of fuzzy numbers or trapezoidal approximations of fuzzy numbers are all using piecewise linear functions to approximate fuzzy numbers. Such approximations are indeed simple and practical enough as the used piecewise linear functions have no knot or have only one knot. However, with the increase of the number of the knots, such approximations will become more and more complex. It brings difficulties for working out the approximation with multiple nodes of the fuzzy number. In the fuzzy control, threshold method is often used. Actually, the threshold method is to use simple threshold function to representation of fuzzy numbers. From this inspiration, in this paper, we study the simple fuzzy number approximations in membership functions space, which is still simple enough and flexible to get the nearest (with respect to metric d) approximation (see Theorems 4 and 5) of a fuzzy number as the number of knots gets large. In Section 2, we briefly review some basic notions, definitions and results about discrete fuzzy numbers. In Section 3, we give the concept of simple fuzzy numbers (which is a special kind of fuzzy numbers with simple structure), obtain the expression of the membership function of thus fuzzy numbers, and prove that a fuzzy set of real number field R is a simple fuzzy number if and only if it is a normal step type fuzzy set. In Section 4, we show that simple fuzzy number space is dense in the fuzzy number space with respect to the metric d (see the Section 2), and obtain two theorems about approximations of fuzzy number membership functions by using simple fuzzy number membership functions. In Section 5, we investigate the properties of approximation operators given by us. In Section 6, we make a brief summary to this paper.

2. Basic definitions and notations

A fuzzy subset (in short, a fuzzy set) of the real line R is a function $u : R \to [0, 1]$. For each such fuzzy set u, we denote by $[u]^r = \{x \in R : u(x) \geq r\}$ for any $r \in (0, 1]$, its r-level set. By supp$u$ we denote the support of u, i.e., the $\{x \in R : u(x) > 0\}$. By $[u]^0$ we denote the closure of the supp$u$, i.e., $[u]^0 = \overline{\{x \in R : u(x) > 0\}}$.

If u is a normal and fuzzy convex fuzzy set of R, $u(x)$ is upper semi-continuous, and $[u]^0$ is compact, then we call u a fuzzy number, and denote the collection of all fuzzy numbers by E.

It is known that if $u \in E$, then for each $r \in [0, 1]$, $[u]^r$ is a convex compact set in R, i.e., a closed interval. For $u \in E$, we denote the closed interval as $[u]^r = [\underline{u}(r), \overline{u}(r)]$ for any $r \in [0, 1]$.

For $a \in R$, define fuzzy number $\hat{a}$ as following:

$$\hat{a}(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}$$

For any $u, v \in E$, define $u \leq v$ if and only if $[u]^r \leq [v]^r$, i.e., $\underline{u}(r) \leq \underline{v}(r)$ and $\overline{u}(r) \leq \overline{v}(r)$ for any $r \in [0, 1]$.

If mapping $d : E \times E \to R$ is defined by

$$d(u, v) = \sqrt{\int_0^1 (\underline{u}(r) - \underline{v}(r))^2 dr + \int_0^1 (\overline{u}(r) - \overline{v}(r))^2 dr}$$

for any $(u, v) \in E \times E$, then d satisfies (see [9])
(1) $d(u, v) \geq 0$ for any $u, v \in E$, and $d(u, v) = 0 \Longleftrightarrow u = v$;
(2) $d(u, v) = d(v, u)$ for any $u, v \in E$;
(3) $d(u, v) \leq d(u, w) + d(w, v)$ for any $u, v, w \in E$,
i.e., d is a metric in E.

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