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Fuzzy differential equations with interactive derivative

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Abstract

In this paper we introduce and study new concept of differentiability for fuzzy-set-valued functions. This derivative considers possible local interactivity in the process studied. Several properties of differentiability and integrability are investigated for the new concept and they are compared to similar fuzzy differentiabilities like Hukuhara differentiability and generalized Hukuhara differentiability. Furthermore, we establish theorems as the fundamental theorem of calculus. Ultimately, we exhibit some results for fuzzy initial value problem and an application.

Keywords: Analysis; Fuzzy-set-valued function; Interactive fuzzy process; Fuzzy derivative; Fuzzy differential equation

1. Introduction

There are at least two types of theory of fuzzy differential equations (FDEs) in the literature. The first uses the derivative for fuzzy-set-valued functions (also known as fuzzy processes) where a fuzzy function associates a fuzzy number with each $t \in [a, b]$. Such derivatives were introduced by Puri and Ralescu [1] and they originated from the Hukuhara derivative for real set-valued functions. The theory of fuzzy differential equations derived from this derivative is the most widespread and has been widely studied [2–6]. This type of differential equation suffers from the defect of having solutions with increasing diameters over time [2,3,7]. This means that as the time passes, the more fuzzy (diffuse, uncertain) the process becomes. Bede and Gal in [8] improved on the concept of derivative presented by Puri and Ralescu in [1] in such a way that the solutions of FDEs do not have, necessarily, increasing diameter. That is, the process can become less fuzzy in the course of the time. The second type of theory of FDEs makes use of fuzzy sets of functions instead of fuzzy-set-valued functions. We consider the fuzzy sets of functions as fuzzy functions, though they are not functions, strictly speaking. These were introduced by Hullermeier in [9] (in parallel by Baidosov [10]) whose theoretical basis is the theory of differential inclusions and fuzzy differential inclusions (see [11]). In this type of FDEs, there is no concept of derivative of fuzzy functions. The derivative used is the same as that adopted for standard functions since it differentiates the functions of the support of fuzzy sets of functions.

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Recently, for this type of fuzzy function, Barros et al. [12] introduced the concept of derivative from the fuzzification operator "derivation" initially applied to real-valued functions. Thus, they present, in fact, a theory of FDE that is very close to the approach of [9] and [10].

The FDEs studied in this paper are developed for fuzzy-number-valued functions, that is, fuzzy functions of type $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$, where $\mathbb{R}_{\mathcal{F}}$ is the space fuzzy numbers. Nevertheless, the derivative is based on the difference between fuzzy sets, that is, the difference is obtained from possibility (or membership) distributions of the fuzzy sets involved, considering its joint possibility distributions [13].

The derivative we study takes into account possible interactivities (dependencies) present in the process to be studied. Therefore, in this methodology, the "infinitesimal" takes into account, that is, incorporates, possible correlations in the process according to the joint possibility distributions involved. As a result, the derivative of fuzzy processes $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ changes from case to case, depending on the joint possibility distribution adopted. This is the fundamental difference from the case the derivative of fuzzy process is the Hukuhara derivative (or generalized [8]), initially defined for classic multi-valued function and where the interactivity (dependence) issues are not present (assumed to be independent or are neglected).

In our proposal, the α-levels of the derivative are expressed according to a joint possibility distribution adopted a priori, while in the case of the Hukuhara derivative (and its generalizations), the α-levels are determined a priori according to the Hukuhara difference. In this last case, the "interactivity" is prefixed and this presupposes that all the fuzzy process studied by this method has the same type of "correlation". As we shall see, the fuzzy differential equations via derivative Hukuhara is a particular case of FDE using our derivative with interactivity.

We will focus our attention on the decay problem

$$\begin{cases} u' &= -\lambda u \\ u(0) &= u_0 \end{cases} \tag{1}$$

and where we obtain the solution $u(t) = u_0 e^{-\lambda t}$ with $u_0 \in \mathbb{R}_{\mathcal{F}}$ and $\lambda > 0$. Moreover, we will argue that it is inconsistent to adopt the Hukuhara derivative for the study of any problem of decay. According to our point of view, the FDE from the Hukuhara derivative perspective can be adopted just in *expansive processes*. Also, we will show that the derivative of the fuzzy function $F(x) = c.g(x)$ is $F'(x) = c.g'(x)$ for $g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}_{\mathcal{F}}$, when g is continuously differentiable as Bede and Gal [8] have also done.

2. Preliminary

A fuzzy subset A of $\mathbb{R}^n$ is given by its membership function $\mu_A : \mathbb{R}^n \longrightarrow [0, 1]$, where $\mu_A(x)$ means the degree to which x belongs to A. The α-levels of the fuzzy subset A are defined as:

$$[A]_\alpha = \{x \in \mathbb{R}^n : \mu_A(x) \geq \alpha\} \text{ for } 0 < \alpha \leq 1 \text{ and}$$
$$[A]_0 = \overline{\{x \in \mathbb{R}^n : \mu_A(x) > 0\}} \text{ for } \alpha = 0.$$

The fuzzy subset A of $\mathbb{R}$ is a fuzzy number if all their α-levels are closed and nonempty intervals of $\mathbb{R}$ and the support of A, $supp(A) = \{x \in \mathbb{R} : \mu_A(x) > 0\}$, is finite. The family of the fuzzy subsets of $\mathbb{R}^n$ with nonempty compact and convex α-levels is denoted by $\mathcal{F}(\mathbb{R}^n)$, while the family of fuzzy numbers is denoted by $\mathbb{R}^n_{\mathcal{F}}$.

The Pompeiu–Hausdorff distance $d_\infty : \mathbb{R}^n_{\mathcal{F}} \times \mathbb{R}^n_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$, is defined by

$$d_\infty(A, B) = \sup_{0 \leq \alpha \leq 1} d_H([A]_\alpha, [B]_\alpha) \tag{2}$$

where d_H is the Pompeiu–Hausdorff distance for sets in $\mathbb{R}^n$. If A and B are fuzzy numbers, that is, $A, B \in \mathbb{R}_{\mathcal{F}}$, then (2) becomes

$$d_\infty(A, B) = \sup_{0 \leq \alpha \leq 1} \max\{|a^-_\alpha - b^-_\alpha|, |a^+_\alpha - b^+_\alpha|\}.$$

From now on, when we refer to a continuous fuzzy function we mean it is continuous in relation to metric d_∞. We denote by $+$ and $-$ the traditional (Minkowski) sum and difference between fuzzy numbers which can be also defined via Zadeh's extension principle [14].

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