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Symmetric difference operators on fuzzy sets ☆

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Abstract

Based on the properties of symmetric difference of sets, a symmetric difference operator for fuzzy sets is defined to be a continuous and associative binary operator on the closed unit interval with some boundary condition. Structures and properties of these operators are investigated in this paper. The main results are: (1) It is proved that a symmetric difference operator is determined by a continuous t-conorm and a strong negation operator on the unit interval. (2) Two models of these operators are discussed. These models are related to the solutions of certain functional equations on the unit interval. In particular, the results presented here provide a partial answer to a problem raised by Alsina, Frank and Schweizer in 2003 about functional equations on the unit interval.

Keywords: Symmetric difference operator; t-Norm; t-Conorm; Strong negation; Associativity

1. Introduction

Given a set X, the symmetric difference operator on the powerset 2^X refers to the binary operator $\triangle : 2^X \times 2^X \to 2^X$ given by

$$A\triangle B = \{x \mid \text{ either } x \in A \text{ but } x \notin B, \text{ or } x \in B \text{ but } x \notin A\}.$$

The operator $\triangle$ has the following properties: for all $A, B, C \in 2^X$,

(a) $A\triangle B = B\triangle A$, (symmetry)
(b) $\emptyset\triangle A = A\triangle\emptyset = A$,
(c) $X\triangle A = A\triangle X = X \setminus A$,
(d) $(A\triangle B)\triangle C = A\triangle(B\triangle C)$, (associativity)
(e) $A\triangle A = \emptyset$.

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Symmetric difference is a basic operation on sets. So, since the introduction of fuzzy sets by Zadeh [26], it has been an interesting topic to extend this operation to the fuzzy setting, see e.g. [1,7,9,10,14,15,18,20,22].

The symmetric difference operator on the powerset 2^X is clearly determined by the function

$$\triangle : \{0,1\} \times \{0,1\} \to \{0,1\}$$

given by

$$\triangle(0,0) = \triangle(1,1) = 0, \quad \triangle(0,1) = \triangle(1,0) = 1.$$

Analogously, a symmetric difference operator on the fuzzy powerset $[0,1]^X$ reduces to an operator

$$\triangle : [0,1] \times [0,1] \to [0,1]$$

that satisfies certain conditions.

There are two approaches to the study of symmetric difference operators on fuzzy sets. The first is axiomatic, the second is model theoretic. Both approaches are based, of course, on the symmetric difference between crisp sets.

The axiomatic approach starts from the properties (a)–(e) of the symmetric difference between crisp sets. It would be desirable if one could find functions $\triangle : [0,1] \times [0,1] \to [0,1]$ with the following conditions: for $x, y, z \in [0,1]$,

(i) $\triangle(x, y) = \triangle(y, x)$, (symmetry)
(ii) $\triangle(0, x) = \triangle(x, 0) = x$,
(iii) $\triangle(1, x) = \triangle(x, 1) = n(x)$,
(iv) $\triangle(\triangle(x, y), z) = \triangle(x, \triangle(y, z))$, (associativity)
(v) $\triangle(x, x) = 0$,

where n is a strong negation on $[0,1]$. But, as shown by Alsina and Trillas [7], there does not exist a function that satisfies the conditions (iii)–(v). Thus, in the study of symmetric difference operators, one has to drop or weaken some of the requirements listed in (i)–(v).

In [7], dropping associativity, Alsina and Trillas defined a symmetric difference operator to be a function $\triangle : [0,1] \times [0,1] \to [0,1]$ that satisfies the conditions (ii), (iii), and (v). In 2013, Dombi [14] investigated the properties of these operators, and proposed a new definition of symmetric difference operators by modifying (v). The articles [10,20] are also concerned with such operators.

In [9], Bedregal, Reiser, and Dimuro defined a symmetric difference operator to be a function $\triangle : [0,1] \times [0,1] \to [0,1]$ that satisfies (i), (ii), (iv), and the boundary condition $\triangle(1,1) = 0$ (a weakening of (v)). The articles [18,20] continued the study of these operators.

The model theoretic approach is based on the following formulas for symmetric difference between crisp sets: for any $A, B \subseteq X$,

$$A \triangle B = (A \cap (X \setminus B)) \cup (B \cap (X \setminus A)),$$
$$A \triangle B = (A \cup B) \cap (X \setminus (A \cap B)).$$

The analogue of these formulas in fuzzy set theory are respectively

$$S(T(x, n(y)), T(n(x), y))$$

and

$$T(S(x, y), nT(x, y)),$$

where, S is a continuous t-conorm, T is a continuous t-norm, and n is a strong negation on $[0,1]$. It is clear that functions of the form $S(T(x, n(y)), T(n(x), y))$ or $T(S(x, y), nT(x, y))$ satisfy the conditions (i), (ii), (iii), and the boundary condition $\triangle(1,1) = 0$. So, they have been studied as natural candidates for symmetric difference operators in fuzzy set theory, see e.g. [1,3,4,9,19]. Then, it is natural to ask: when are they associative? The reader is referred to [3,4,9,18,20] for more on this topic. Relevant works can also be found in [2,5,6,8,11–13].

In this article, being aware of the importance of the associativity of a binary operator, we are concerned with symmetric difference operators that satisfy the associative law. But, unlike [9], we define a symmetric difference operator to be a continuous function $\triangle : [0,1] \times [0,1] \to [0,1]$ that satisfies (ii), (iv), and the boundary condition $\triangle(1,1) = 0$.

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